

LINKED SYMPLECTIC FORMS AND LIMIT LINEAR SERIES IN RANK 2 WITH SPECIAL DETERMINANT

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ABSTRACT. We generalize the prior linked symplectic Grassmannian construction, applying it to prove smoothing results for rank-2 limit linear series with fixed special determinant on chains of curves. We apply this general machinery to prove new results on nonemptiness and dimension of rank-2 Brill-Noether loci in a range of degrees.

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1. INTRODUCTION

At its most basic, higher-rank Brill-Noether theory addresses the question: for a general curve of genus g , how many global sections can a (semi)stable vector bundle of given rank and degree have? This has been an active subject of study for more than 20 years, and the picture which has emerged is one of complexity, with no comprehensive conjectures even in the case of rank 2. Nonetheless, those cases which have been understood have already been important in a number of strikingly different contexts, from Mukai's work [Muk01] on classification of Fano threefolds to recent work of Bhosle, Brambila-Paz and Newstead [BBPN] on Butler's conjecture. See [GT09] and the introduction of [Oss14a] for a more detailed survey.

In the present paper, we consider $\mathbf{g}_{2,d}^k$ s consisting of pairs (\mathcal{E}, V) , where \mathcal{E} is a vector bundle of rank 2 and degree d , and V is a k -dimensional space of global sections, and we extend the range where stable $\mathbf{g}_{2,d}^k$ in a range of degrees and

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genera. Our main tool for doing this is a new smoothing theorem in the case of special determinant, and as in the classical rank-1 case treated by Eisenbud and Harris, our smoothing theorem is inseparable from a careful study of dimension. Now, the classical expected dimension ρ generalizes to higher rank, and in some cases gives the correct dimension for the moduli space of $\mathfrak{g}_{r,d}^k$ s, but is not enough to understand the general picture. One case where the picture seems clearer is that of rank-2 vector bundles with fixed canonical determinant: Bertram, Feinberg [BF98] and Mukai [Muk95] observed that in this case, the behavior appears to be closer to that of the classical rank-1 case, albeit with a modified expected dimension. While the naive expected dimension for the fixed determinant case is $\rho - g$, they showed that symmetries in the canonical determinant case forced the dimension to be at least $\rho_\omega := \rho - g + \binom{k}{2}$, and they conjectured that this is in fact the correct dimension, and in particular that the relevant moduli spaces should be nonempty when $\rho_\omega \geq 0$. The existence portion of their conjecture remains open, while their results on modified expected dimension were generalized by the first author to a wider class of special determinants in [Oss13a].

In [Tei91a], the second author developed a theory of limit linear series in higher rank, and in [Tei04] applied it to show that, subject to proving a certain smoothing theorem, one could use it to prove existence for a large infinite family of cases of the Bertram-Feinberg-Mukai conjecture. The main obstruction to proving the smoothing theorem is that the symmetries which occur in the canonical determinant case and lead to the larger expected dimension do not persist in an obvious way in the definition of higher-rank limit linear series. In [OT14], we showed that by relating higher-rank limit linear series to an alternate limit linear series construction first introduced in [Oss06], it is possible to prove the necessary smoothing theorem for curves having two irreducible components. However, because higher-rank limit linear series do not have a simple inductive structure, this is not enough to prove the desired existence results. The purpose of the present paper, in conjunction with [Oss14a], is to generalize the constructions of [OT14] to work on chains of curves with any number of components, leading to the proof of the desired smoothing theorem. In addition, we broaden the families of examples considered in [Tei04] to more general special determinants, consequently proving existence of large families of components of moduli spaces of $\mathfrak{g}_{2,d}^k$ s with fixed special determinant and having the (modified) expected dimension.

We now explain our smoothing theorem for higher-rank limit linear series. Given nonnegative integers g, d, k , the naive expected dimension for the stack of $\mathfrak{g}_{2,d}^k$ s on a general curve of genus g is equal to

$$\rho - 1 = 4g - 4 - k(k - d + 2g - 2).$$

Thus, given a line bundle of degree d on such a curve X , the naive expected dimension for the stack of $\mathfrak{g}_{2,d}^k$ s on X with fixed determinant \mathcal{L} is given by $\rho - g$, but according to [Oss13a] the higher expected dimension $\rho - g + \binom{k}{2}$ applies when \mathcal{L} is special. Obviously, this distinction is only relevant if $k \geq 2$, so we will impose this assumption. A typical smoothing theorem for limit linear series says that if one has a family of limit linear series which occurs in the expected dimension on a given reducible curve, then it smooths out to give linear series on nearby smooth curves. For our smoothing theorem, we restrict to the case that the special fiber is a chain of curves, as has been the case for every family considered in [Tei91a] and [Tei04],

and we work with a certain open subset of “chain-adaptable” (higher-rank) limit linear series, introduced in [Oss14a]. In order to treat (semi)stability conditions more easily, we work with the notion of ℓ -(semi)stability introduced in [Oss14b]. Compared to the usual notion of stability on a reducible curve, this is weaker and more canonical, and leads to equally strong conclusions in degeneration arguments. Our main smoothing theorem is then as follows.

Theorem 1.1. *Given g, d, k , with $k \geq 2$, suppose that there exists a projective nodal curve X_0 with dual graph Γ a chain, a special line bundle \mathcal{L}_0 of degree d on X_0 , and $((\mathcal{E}_v, V_v)_{v \in V(\Gamma)}, (\varphi_e)_{e \in E(\Gamma)}, \psi)$ a chain-adaptable limit linear series of rank 2 and fixed determinant \mathcal{L}_0 , such that the space of such limit linear series on X_0 has the expected dimension $\rho - g + \binom{k}{2}$ at the corresponding point.*

Then for a general smooth curve X of genus g and a general special line bundle \mathcal{L} of degree d , the space of $\mathfrak{g}_{2,d}^k$ s with fixed determinant \mathcal{L} on X is nonempty, with a component of expected dimension $\rho - g + \binom{k}{2}$.

Furthermore, if $((\mathcal{E}_v, V_v)_{v \in V(\Gamma)}, (\varphi_e)_{e \in E(\Gamma)}, \psi)$ is ℓ -semistable (respectively, ℓ -stable), then we get the same statement for semistable (respectively, stable) $\mathfrak{g}_{2,d}^k$ s on X .

The relevant limit linear series terminology is reviewed in Section 7 below.

Theorem 1.1 renders unconditional the existence results in [Tei04] for the case of canonical determinant. Moreover, using similar techniques, we produce more general families of limit linear series with special determinant, and we conclude more general existence results, as follows:

Theorem 1.2. *Given g, d, k nonnegative, with $k \geq 2$ and $g - 2 \leq d \leq 2g - 2$, suppose that*

$$4g \geq \begin{cases} k^2 + 2k(2g - 2 - d) : & d \text{ even} \\ k^2 + 4 + 2k(2g - 2 - d) : & d \text{ odd.} \end{cases}$$

Then for a general smooth curve X of genus g and a general special line bundle \mathcal{L} of degree d , the space of semistable $\mathfrak{g}_{2,d}^k$ s with fixed determinant \mathcal{L} on X is nonempty, with a component of expected dimension $\rho_{\mathcal{L}} := \rho - g + \binom{k}{2}$. If further

$$(g, d, k) \neq (1, 0, 2), (2, 2, 2), (3, 2, 2) \text{ or } (4, 6, 4),$$

the same is true of the space of stable $\mathfrak{g}_{2,d}^k$ s.

Note that we do not have to assume that $\rho_{\mathcal{L}} \geq 0$, and in fact this is not always the case; see Remark 9.2. Also, the case $(g, d, k) = (2, 2, 2)$ is an exception to the existence of stable bundles in the Bertram-Feinberg-Mukai conjecture: although $\rho_{\mathcal{L}} = 0$ in this case, there cannot be any stable $\mathfrak{g}_{2,2}^2$. See the note on page 123 of [Tei91b]. In particular, within the imposed range for g , our stability results are optimal.

This existence result can be seen as validation of the first main result of [Oss13a], insofar as it provides many examples of components of moduli spaces of $\mathfrak{g}_{2,d}^k$ s with fixed special determinant having dimension equal to the modified expected dimension $\rho - g + \binom{k}{2}$. However, Theorem 1.1 sets up a more general machinery, and indeed Zhang [Zha14] has already been able to use it to prove existence results for the canonical determinant case which improve on those of Theorem 1.2. In this context, we also mention recent work of Lange, Newstead and Park [LNP] which

approaches the existence question in the canonical determinant case via fundamental class computations. Their results are comparable to those of Zhang, except that due to combinatorial complications, they restrict to prime genera.

Finally, by allowing determinants to vary we immediately conclude existence of components of larger than the expected dimension $\rho - 1$ in a wide range of cases.

Corollary 1.3. *Given g, d, k nonnegative, with $k \geq 2$ and $g - 2 \leq d \leq 2g - 2$, suppose that*

$$4g \geq \begin{cases} k^2 + 2k(2g - 2 - d) : & d \text{ even} \\ k^2 + 4 + 2k(2g - 2 - d) : & d \text{ odd.} \end{cases}$$

Then for a general smooth curve X of genus g , the space of semistable $\mathfrak{g}_{2,d}^k$ is nonempty, with a component of dimension $\rho + \binom{k}{2} - (d - g + 3)$. If further

$$(g, d, k) \neq (1, 0, 2), (2, 2, 2), (3, 2, 2) \text{ or } (4, 6, 4),$$

the same is true of the space of stable $\mathfrak{g}_{2,d}^k$.

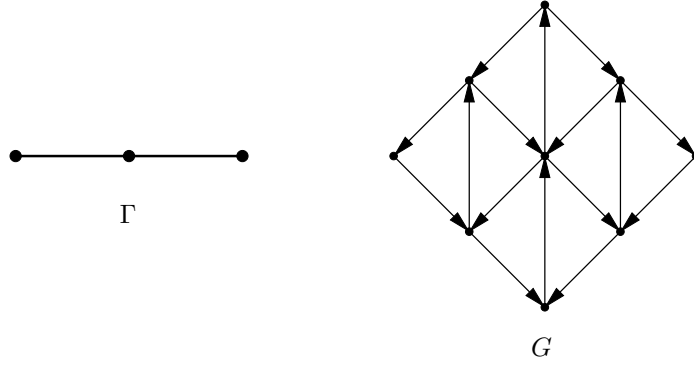
In particular, if $\binom{k}{2} > d - g + 2$, the space has a component of dimension strictly greater than $\rho - 1$.

The main theoretical tool in the proof of Theorem 1.1 involves the definition and study of a generalization of the linked alternating Grassmannian introduced in [OT14], together with an associated nondegeneracy condition which replaces the notion of linked symplectic forms. When combined with the general limit linear series theory developed in [Oss14a], we are able to deduce our general smoothing results. Rather than considering a strict generalization of the linked symplectic Grassmannian of [OT14], we consider a somewhat broader notion; this simplifies definitions and arguments, without weakening the resulting smoothing theorems. In brief, prelinked alternating Grassmannians are cut out inside prelinked Grassmannians by an isotropy condition with respect to a “linked alternating form.” Prelinked Grassmannians have an open subset consisting of “simple points,” and we can analyze the behavior of prelinked alternating Grassmannians on this subset. We also introduce a smaller subset of “internally simple points,” and if the prelinked alternating form in question is “internally symplectic,” we get especially good behavior, which leads to our smoothing theorem. The relevant statement is as follows.

Theorem 1.4. *If $\text{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet) \subseteq \text{LG}(r, \mathcal{E}_\bullet)$ is a prelinked alternating Grassmannian, and $z \in \text{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is a simple point of $\text{LG}(r, \mathcal{E}_\bullet)$, then locally at z , we have that $\text{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is cut out by $\binom{r}{2}$ equations inside $\text{LG}(r, \mathcal{E}_\bullet)$.*

If further \langle, \rangle is internally symplectic, and z is an internally simple point, then $\text{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is smooth at z of codimension $\binom{r}{2}$ inside $\text{LG}(r, \mathcal{E}_\bullet)$.

In generalizing the ideas of [OT14] to the present situation, the main challenge is to determine the suitable definitions, which are not obvious from the previously developed two-component case. Given the correct definitions, the arguments of [OT14] go through in a transparent manner to obtain Theorem 1.4. For further motivation of our definitions, see [OT14], especially Remarks 3.8 and 4.6.

FIGURE 1. An example of Situation 2.1, with $b = 2$.

2. PRELIMINARIES

We begin with some definitions of a combinatorial nature. Ultimately, Γ will be the dual graph of a reducible curve, and the associated graph G will be used to keep track of multidegrees of vector bundles and natural maps between them.

Situation 2.1. Let Γ be a tree. Let d, k be positive integers, and fix also integers b and d_v for each $v \in V(\Gamma)$, satisfying

$$\sum_{v \in V(\Gamma)} d_v - |E(\Gamma)|2b = d.$$

We define a directed graph G as follows: let $V(G) \subseteq \mathbb{Z}^{V(\Gamma)}$ consist of vectors $w = (i_v)_{v \in V(\Gamma)}$ satisfying:

- (i) $\sum_v i_v = d$,
- (ii) $i_v \equiv d_v \pmod{2}$ for all $v \in V(\Gamma)$,
- (iii) for every subtree Γ' of Γ obtained as a connected component of the complement of some edge of Γ ,

$$|V(\Gamma')|2b - \sum_{v \in \Gamma'} (d_v - i_v) \geq 0.$$

Then, G has an edge from w to w' in $V(G)$ if there is a vertex $v \in V(\Gamma)$ of valence ℓ such that $w' - w$ is -2ℓ in index v , is 2 in index v' for each v' adjacent to v , and is 0 elsewhere. Given an edge $\varepsilon \in E(G)$, let $v(\varepsilon)$ be the associated vertex of $V(\Gamma)$.

We also define the hyperplane $H_d \subseteq \mathbb{Z}^{V(\Gamma)}$ to be the set of vectors satisfying (i) above.

Note that given $w, w' \in V(G)$, we have $\frac{w+w'}{2} \in H_d$.

Without further comment, we will assume that we are in the above situation, and refer freely to the above notation.

It is convenient to introduce a distance function on H_d as follows:

Definition 2.2. Given $w, w' \in H_d$, define $d(w, w')$ to be the minimal number of operations required to get from w to w' , where an allowable operation is, for a choice of $v \in V(\Gamma)$ of valence ℓ , to subtract ℓ from the v th coordinate and add 1 to the v' th coordinate for each vertex v' adjacent to v .

Remark 2.3. Situation 2.1 is described more generally in Definitions 3.1.2 and 3.4.9 of [Oss14a] (where G is called \tilde{G}_{II}), with an additional parameter r which in our case is set to 2. From this point of view, $d(w, w')$ is in essence the shortest path from w to w' in G , considered for the case $r = 1$, except without the imposition of condition (iii) for the vertices of G . Note however that due to the directed nature of G , this distance function is not symmetric.

We now move on to the fundamental definitions. We work throughout over a fixed base scheme S . We will use h and t to denote the head and tail of an edge of a graph, or a path in a graph.

Definition 2.4. Suppose we are given data \mathcal{E}_\bullet consisting of vector bundles \mathcal{E}_w of rank 2 on S for each $w \in V(G)$, and morphisms $f_\varepsilon : \mathcal{E}_{t(\varepsilon)} \rightarrow \mathcal{E}_{h(\varepsilon)}$ for each $\varepsilon \in E(G)$. Given a directed path $P = (\varepsilon_1, \dots, \varepsilon_n)$ in G , denote by f_P the composition $f_{\varepsilon_n} \circ \dots \circ f_{\varepsilon_1}$, and by $v(P)$ the set (considered with multiplicities) $\{v(\varepsilon_1), \dots, v(\varepsilon_n)\}$.

Given also $s \in \Gamma(S, \mathcal{O}_S)$, we say that \mathcal{E}_\bullet is **s -prelinked** if for P and P' any directed paths in G with the same tail and head and with the length of P less than or equal to the length of P' , we have

$$f_{P'} = s^c f_P,$$

where c is the difference between the number of times $V(\Gamma)$ appears in $v(P')$ and the number of times $V(\Gamma)$ appears in $v(P)$.

This definition is slightly more restrictive than that considered in Appendix A of [Oss14a], reflecting that we consider here a more specific collection of graphs G onto which we may impose more structure.

Note that for two given vertices of G , any two minimal paths P, P' have $v(P) = v(P')$, and more generally, any path P has the same $v(P)$ as a path obtained as a minimal path together with insertions of copies of $V(\Gamma)$ – see Proposition 3.1.4 of [Oss14a]. Thus, the condition implies in particular that if P is a minimal path between two vertices, the resulting map f_P does not depend on P , and more generally it specifies precisely how f_P depends on P . Thus, the following notation makes sense:

Notation 2.5. If \mathcal{E}_\bullet is s -prelinked, given $w, w' \in V(G)$, we denote by $f_{w, w'} : \mathcal{E}_w \rightarrow \mathcal{E}_{w'}$ the map f_P determined by any minimal path P from w to w' .

Definition 2.6. Let K be a field over S , and for $s \in K$, let E_\bullet be s -prelinked on $\text{Spec } K$. We say that E_\bullet is **simple** if there exist $w_1, \dots, w_r \in V(G)$ (not necessarily distinct) and $v_i \in E_{w_i}$ for $i = 1, \dots, r$ such that for every $w \in V(G)$, there exist paths P_1, \dots, P_g with each P_i going from w_i to w , and such that $f_{P_1}(v_1), \dots, f_{P_r}(v_r)$ form a basis for E_w .

The following is a straightforward consequence of Nakayama's lemma.

Proposition 2.7. *Suppose that \mathcal{E}_\bullet is simple at a point $x \in S$. Then locally at x , for $w \in G$ there exist $r_w \geq 0$ and subbundles $\mathcal{W}_w \subseteq \mathcal{E}_w$ of rank r_w such that:*

- (i) $\sum_{w \in G} r_w = r$.
- (ii) *The natural map*

$$\bigoplus_{w' \in G} f_{w', w}(\mathcal{W}_{w'}) \rightarrow \mathcal{E}_w$$

is an isomorphism for each $w \in G$.

In particular, the points of S at which \mathcal{E}_\bullet is simple form an open subset.

3. LINKED BILINEAR FORMS

In this section and the next, we investigate the new definitions which form the basis for our foundational results. In some sense, the definitions are the most important part, as they are calibrated so that the arguments already used in [OT14] will go through in the more general setting. See Remarks 3.8 and 4.6 and Examples 5.3 and 5.4 of [OT14] for discussion of the motivation behind the definitions.

As we discuss linked bilinear forms, we restrict from now on to the case $r = 2$. For arbitrary r , one can work instead with multilinear forms, but nondegeneracy conditions seem much harder to understand in this context; compare to [Oss13c].

Definition 3.1. Given an s -prelinked \mathcal{E}_\bullet , and $m \in H_d$, a **linked bilinear form of index m** on \mathcal{E}_\bullet is a collection of bilinear pairings for each $w, w' \in V(G)$

$$\langle, \rangle_{w, w'} : \mathcal{E}_w \times \mathcal{E}_{w'} \rightarrow \mathcal{O}_S$$

satisfying the following compatibility conditions: for any suitable w, w' , and $\varepsilon \in E(G)$ with $t(\varepsilon) = w$, we have

$$\langle, \rangle_{h(\varepsilon), w'} \circ (f_\varepsilon \times \text{id}) = s^{\delta_{w, w', \varepsilon}} \cdot \langle, \rangle_{w, w'},$$

and for any ε' with $t(\varepsilon') = w'$, we have

$$\langle, \rangle_{w, h(\varepsilon')} \circ (\text{id} \times f_{\varepsilon'}) = s^{\delta_{w, w', \varepsilon'}} \cdot \langle, \rangle_{w, w'},$$

where

$$\delta_{w, w', \varepsilon} = \begin{cases} 1 : d(\frac{w+w'}{2}, m) < d(\frac{h(\varepsilon)+w'}{2}, m) \\ 0 : \text{otherwise,} \end{cases}$$

and

$$\delta_{w, w', \varepsilon'} = \begin{cases} 1 : d(\frac{w+w'}{2}, m) < d(\frac{w+h(\varepsilon')}{2}, m) \\ 0 : \text{otherwise,} \end{cases}.$$

Note that if $w = w'$, so that the notation $\delta_{w, w', \varepsilon}$ is *a priori* ambiguous, we have that both definitions coincide, so in fact there is no ambiguity.

The following lemma checks that our compatibility conditions are internally consistent, and will be useful later. The notation is slightly cumbersome, because we have to keep track of the order of applying maps on the left and the right, but the idea is simply to see what our compatibility conditions say when we apply sequences of maps on either side.

Lemma 3.2. *Given $w, w' \in V(G)$, and paths P and P' starting at w, w' respectively, with $P = (\varepsilon_1, \dots, \varepsilon_n)$ and $P' = (\varepsilon_{n+1}, \dots, \varepsilon_{n+n'})$, fix $\sigma \in S_{n+n'}$ such that σ^{-1} preserves the order of $\{1, \dots, n\}$ and $\{n+1, \dots, n+n'\}$. For $i = 1, \dots, n+n'$, set*

$$f_i = \begin{cases} f_{\varepsilon_{\sigma(i)}} \times \text{id} : & \sigma(i) \leq n \\ \text{id} \times f_{\varepsilon_{\sigma(i)}} : & \sigma(i) > n, \end{cases}$$

and let

$$(3.1) \quad \delta_{P, P'} = \frac{1}{|V(\Gamma)|} \left(n + n' + d\left(\frac{h(P) + h(P')}{2}, m\right) - d\left(\frac{w + w'}{2}, m\right) \right).$$

Then the compatibility conditions imposed in Definition 3.1 give

$$\langle, \rangle_{h(P), h(P')} \circ f_{n+n'} \circ \dots \circ f_1 = s^{\delta_{P, P'}} \cdot \langle, \rangle_{w, w'},$$

In fact, if we denote by P_i for $i = 1, \dots, n + n'$ the truncation of P given by $(\varepsilon_j, \dots, \varepsilon_n)$ where j is minimal in $\{1, \dots, n\}$ with $\sigma^{-1}(j) \geq i$, and similarly for P'_i , we have

$$\sum_{i=1}^{n+n'} \delta_{t(P_i), t(P'_i), \varepsilon_{\sigma(i)}} = \delta_{P, P'}.$$

We use the convention that if P_i is empty, then $t(P_i) = h(P)$, and similarly for P'_i .

Proof. Setting $w_i = t(P_i)$ and $w'_i = t(P'_i)$ for each i , the value of each $\delta_{t(P_i), t(P'_i), \varepsilon_{\sigma(i)}}$ is determined by $d(\frac{w_{i+1} + w'_{i+1}}{2}, m) - d(\frac{w_i + w'_i}{2}, m)$. According to our description of minimal paths, this difference is always either -1 or $|V(\Gamma)| - 1$, and $\sum_{i=1}^{n+n'} \delta_{w_i, w'_i, \varepsilon_{\sigma(i)}}$ can be expressed as the number of times it is $|V(\Gamma)| - 1$. Then the sum of these differences is equal to $(|V(\Gamma)| \sum_{i=1}^{n+n'} \delta_{w_i, w'_i, \varepsilon_{\sigma(i)}}) - n - n'$, giving the desired formula. \square

Definition 3.3. In the notation of Definition 3.1, a linked bilinear form is a **linked alternating form** if $\langle, \rangle_{w, w}$ is an alternating form on \mathcal{E}_w for all w , and

$$\langle, \rangle_{w, w'} = -\langle, \rangle_{w', w} \circ \text{sw}_{w, w'}$$

for all $w \neq w'$, where $\text{sw}_{w, w'} : \mathcal{E}_w \times \mathcal{E}_{w'} \rightarrow \mathcal{E}_{w'} \times \mathcal{E}_w$ is the canonical map switching factors.

This definition is equivalent to requiring that the induced form on $\bigoplus_w \mathcal{E}_w$ be alternating.

Because being s -prelinked is preserved by base change, it makes sense to define moduli functors of linked bilinear forms and linked alternating forms. Moreover, it is clear that these functors are represented by schemes (compare Lemma 2.2 of [Oss11]), and that they further have natural module structures (i.e., for each S -scheme T the set of linked bilinear/alternating forms is a $\Gamma(T, \mathcal{O}_T)$ -module). Our first result is that for simple s -prelinked bundles, the moduli of linked bilinear forms and of linked alternating forms behave just like their classical counterparts, in the sense that the representing scheme is in fact a vector bundle of the appropriate rank.

Proposition 3.4. Suppose $\mathcal{E}_\bullet = ((\mathcal{E}_w)_w, (f_\varepsilon)_\varepsilon)$ is s -prelinked of rank r , and simple, and $m \in H_d$. Then the moduli scheme of linked bilinear forms on \mathcal{E}_\bullet of index m is a vector bundle on S of rank r^2 , and the moduli scheme of linked alternating forms on \mathcal{E}_\bullet of index m is a vector bundle on S of rank $\binom{r}{2}$.

Proof. First, choose subbundles $\mathcal{W}_w \subseteq \mathcal{E}_w$ as provided by Proposition 2.7. Clearly, a linked bilinear form on \mathcal{E}_\bullet induces by restriction a collection of bilinear pairings

$$\langle, \rangle'_{w, w'} : \mathcal{W}_w \times \mathcal{W}_{w'} \rightarrow \mathcal{O}_S,$$

or equivalently a bilinear form on $\bigoplus_w \mathcal{W}_w$, and our claim is that this restriction map induces an isomorphism of functors from linked bilinear forms to bilinear forms on $\bigoplus_w \mathcal{W}_w$. Because $\sum_w \text{rk } \mathcal{W}_w = r$, the claim yields the first statement of the proposition.

To prove the claim, suppose we have a collection of $\langle, \rangle'_{w,w'}$ as above; we aim to construct an inverse to the restriction map. Because

$$\mathcal{E}_w \cong \bigoplus_{w' \in G} f_{w',w}(\mathcal{W}_{w'}),$$

in order to define $\langle v_1, v_2 \rangle_{w,w'}$, it is enough to do so for v_1 in $f_{u,w}(\mathcal{W}_u)$ and v_2 in $f_{u',w'}(\mathcal{W}_{u'})$ as u, u' vary in $V(G)$. Starting from the necessity of having $\langle, \rangle_{w,w'} = \langle, \rangle'_{w,w'}$ on $\mathcal{W}_w \times \mathcal{W}_{w'}$, we then see that inductive application of the compatibility conditions of Definition 3.1 determine $\langle, \rangle_{w,w'}$ uniquely: specifically, if $v \in \mathcal{W}_u$ and $v' \in \mathcal{W}_{u'}$, then we set

$$\langle f_{u,w}(v), f_{u',w'}(v') \rangle_{w,w'} = s^{\delta_{P,P'}} \langle v, v' \rangle_{w,w'},$$

where P and P' are minimal paths from u to w and from u' to w' respectively, and $\delta_{P,P'}$ are as in Lemma 3.2. It remains to check that the result forms $\langle, \rangle_{w,w'}$ satisfy the conditions for a linked bilinear form, which is straightforward to verify directly, again using Lemma 3.2. The claim then follows, as the preceding construction is visibly inverse to the restriction map.

To obtain the second statement of the proposition, it is enough to observe that under the isomorphism of functors constructed above, a linked bilinear form is alternating if and only if the induced form on $\bigoplus_w \mathcal{W}_w$ is alternating. Indeed, this follows from the symmetry of the compatibility conditions together with Lemma 3.2. \square

Definition 3.5. If \mathcal{E}_\bullet is s -prelinked, with a linked bilinear form \langle, \rangle_\bullet , the **locus of isotropy** of \langle, \rangle_\bullet on \mathcal{E}_\bullet is the closed subscheme of S representing the functor of morphisms $T \rightarrow S$ such that \langle, \rangle_\bullet is identically zero after restriction to T .

The fact that the locus of isotropy is represented by a closed subscheme is clear, as \mathcal{E}_\bullet together with \langle, \rangle_\bullet induces a morphism from S to the moduli scheme of linked bilinear forms on \mathcal{E}_\bullet , and the locus of isotropy is the preimage under this morphism of the zero form.

Proposition 3.4 thus implies:

Corollary 3.6. *Suppose \mathcal{E}_\bullet is s -prelinked and simple, and \langle, \rangle_\bullet is a linked bilinear (respectively, linked alternating) form on S . Then the locus of isotropy is locally cut out in S by d^2 (respectively, $\binom{d}{2}$) equations, and thus if S is locally Noetherian, every component of this locus has codimension at most d^2 (respectively, $\binom{d}{2}$) in S .*

4. LINKED SYMPLECTIC FORMS

In [OT14], our definition of linked symplectic form consisted of two nondegeneracy conditions, which we can think of as a condition on the interior of G , and a condition on the boundary. With these conditions, we were able to prove that linked symplectic Grassmannians are smooth of the expected dimension at all simple points. While it is undoubtedly possible to follow the same process in our present generalized setting, we instead simplify both the definition and arguments substantially via the observation that for our purposes (i.e., for proving Theorem 1.1), the boundary is irrelevant, and it is enough to restrict attention to a more restrictive class of “internally” simple points. Thus, our definitions will not precisely generalize those of [OT14], and accordingly we use slightly different terminology.

Definition 4.1. Given an s -prelinked \mathcal{E}_\bullet on a scheme S , and $m \in H_d$, a linked alternating form $(\langle, \rangle_{w,w'})_{w,w' \in V(G)}$ of index m on \mathcal{E}_\bullet is a **linked internally symplectic form** if for each $w, w' \in V(G)$ with $\frac{w+w'}{2} = m$, we have $\langle, \rangle_{w,w'}$ perfect.

Here by perfect, we mean that $\langle, \rangle_{w,w'}$ is nondegenerate after restriction to any point of S .

Definition 4.2. Let K be a field over S , and for $s \in K$, let E_\bullet be s -prelinked on $\text{Spec } K$. Given also $m \in H_d$, we say that E_\bullet is **internally simple** relative to m if there exist $w_1, \dots, w_r \in V(G)$ (not necessarily distinct) and $v_i \in E_{w_i}$ for $i = 1, \dots, r$ such that:

- (I) for every $w \in V(G)$, there exist paths P_1, \dots, P_g with each P_i going from w_i to w , and such that $f_{P_1}(v_1), \dots, f_{P_r}(v_r)$ form a basis for E_w ;
- (II) for each i , we have $2m - w_i \in V(G)$.

It follows from Nakayama's lemma that over an arbitrary base, being internally simple relative to m is an open condition, so we may apply it over any base to mean simply that it holds over each point.

For our main calculation relating to linked symplectic forms, we need to work with respect to subbundles, so we also make the following definition.

Definition 4.3. Given an s -prelinked \mathcal{E}_\bullet , a collection \mathcal{F}_\bullet of subbundles of \mathcal{E}_\bullet of fixed rank r is a **linked subbundle** if for all $e \in G$, from w to w' , we have $f_e(\mathcal{F}_w) \subseteq \mathcal{F}_{w'}$. If also we have a linked alternating form \langle, \rangle_\bullet on \mathcal{E}_\bullet , we say that \mathcal{F}_\bullet is **isotropic** for \langle, \rangle_\bullet if the restriction of \langle, \rangle_\bullet to \mathcal{F}_\bullet vanishes identically.

The following construction arises in the analysis of tangent spaces of prelinked alternating Grassmannians, and is therefore key to the proof of Theorem 1.4.

Definition 4.4. Suppose that \mathcal{E}_\bullet is s -prelinked, and \langle, \rangle_\bullet is a linked alternating form on \mathcal{E}_\bullet of index m . Let $\mathcal{F}_\bullet \subseteq \mathcal{E}_\bullet$ be a linked subbundle which is internally simple relative to m , and suppose that \mathcal{F}_\bullet is isotropic for \langle, \rangle_\bullet . Finally, let $(\mathcal{W}_w \subseteq \mathcal{F}_w)_w$ be as in Lemma 2.7, with the nonzero \mathcal{W}_w supported on $w \in V(G)$ with $2m - w \in V(G)$. Given family of homomorphisms $(\varphi_w : \mathcal{W}_w \rightarrow \mathcal{E}_w / \mathcal{F}_w)_{w \in V(G)}$ define the associated linked alternating form $\langle, \rangle_{\bullet}^{\varphi}$ on \mathcal{F}_\bullet by applying the following formula on the \mathcal{W}_w :

$$\langle, \rangle_{w,w'}^{\varphi} = \langle, \rangle_{w,w'} \circ (\varphi_w \times \text{id}) + \langle, \rangle_{w,w'} \circ (\text{id} \times \varphi_{w'}).$$

Note that this is well-defined because \mathcal{F}_\bullet is assumed to be isotropic. Also, recall that by Proposition 3.4, the pairings on the \mathcal{W}_w defined above uniquely determine a linked alternating form $\langle, \rangle_{\bullet}^{\varphi}$ on \mathcal{F}_\bullet . The main consequence of the symplectic condition is the following.

Lemma 4.5. *In the situation of Definition 4.4, suppose further that \langle, \rangle_\bullet is an internally linked symplectic form, and that S is a point. Then the map from $\bigoplus_{w \in V(G)} \text{Hom}(\mathcal{W}_w, \mathcal{E}_w / \mathcal{F}_w)$ to the space of linked alternating forms on \mathcal{F}_\bullet is surjective.*

We will need a preliminary observation on linked bilinear forms:

Proposition 4.6. *Suppose that \mathcal{E}_\bullet is s -prelinked, and \langle, \rangle_\bullet is a linked bilinear form on \mathcal{E}_\bullet of index m . Given $w, w' \in V(G)$, suppose that $2m - w \in V(G)$. Then, if P is a minimal path from w' to $2m - w$, we have*

$$\langle, \rangle_{w,w'} = \langle, \rangle_{w,2m-w} \circ (\text{id} \times f_P).$$

Proof. Observing that the length of P is equal to $d\left(\frac{w+w'}{2}, m\right)$, this is an immediate consequence of Lemma 3.2. \square

With this, the proof of the lemma is now straightforward.

Proof of Lemma 4.5. If we choose bases v_j^w for each \mathcal{W}_w , it is clearly enough to prove that for all w, u, p, q with either $w \neq u$ or $p \neq q$, there exists a choice of φ_\bullet such that the induced pairing $\langle v_{p'}^{w'}, v_{q'}^{u'} \rangle_{w', u'}^{\varphi_\bullet}$ is nonzero precisely when (w', u', p', q') is either (w, u, p, q) or (u, w, q, p) . Given w, u, p, q , we construct the desired φ_\bullet as follows. First set $\varphi_{w'} = 0$ for all $w' \neq w$, and set $\varphi_w(v_{p'}^w) = 0$ for all $p' \neq p$. We then wish to show that there exists a choice of $\varphi_w(v_p^w) \in \mathcal{E}_w$ such that $\langle \varphi_w(v_p^w), v_q^u \rangle_{w, u} \neq 0$, but $\langle \varphi_w(v_p^w), v_{q'}^{u'} \rangle_{w, u'} = 0$ for all other choices of u', q' . Equivalently, if we denote by $\widehat{\mathcal{W}}_u \subseteq \mathcal{W}_u$ the span of the $v_{q'}^u$ for $q' \neq q$, we want

$$\varphi_w(v_p^w) \in (\widehat{\mathcal{W}}_u)^\perp \cap (\cap_{u' \neq u} \mathcal{W}_{u'}^\perp),$$

but

$$\varphi_w(v_p^w) \notin \cap_{u'} \mathcal{W}_{u'}^\perp.$$

Here each orthogonal space should be taken with respect to the appropriate pairing. Now, by the internally simple hypothesis, we have that $2m - w \in V(G)$, so for each u' , choose a minimal path $P_{u'}$ from u' to $2m - w$. By Proposition 4.6, the above conditions are equivalent to having

$$\begin{aligned} \varphi_w(v_p^w) &\in \left(f_{P_u}(\widehat{\mathcal{W}}_u)\right)^\perp \cap \left(\bigcap_{u' \neq u} (f_{P_{u'}}(\mathcal{W}_{u'}))^\perp\right) \\ &= \left(f_{P_u}(\widehat{\mathcal{W}}_u) \oplus \left(\bigoplus_{u' \neq u} f_{P_{u'}}(\mathcal{W}_{u'})\right)\right)^\perp, \end{aligned}$$

but

$$\begin{aligned} \varphi_w(v_p^w) &\notin \left(\bigcap_{u'} (f_{P_{u'}}(\mathcal{W}_{u'}))^\perp\right) \\ &= \left(\bigoplus_{u'} f_{P_{u'}}(\mathcal{W}_{u'})\right)^\perp. \end{aligned}$$

The sums are direct sums because of Proposition 2.7, and now all the orthogonal complements are relative to $\langle, \rangle_{w, 2m-w}$. Again by Proposition 2.7, the two sums give distinct subspaces of \mathcal{F}_{2m-w} , so by the nondegeneracy of $\langle, \rangle_{w, 2m-w}$ imposed in the definition of a linked internally symplectic form, we conclude that a $\varphi_w(v_p^w)$ satisfying the desired conditions exists, and the lemma follows. \square

5. APPLICATIONS TO LINKED GRASSMANNIANS

We begin with some preliminary background on (pre)linked Grassmannians.

Definition 5.1. Let \mathcal{E}_\bullet be an s -prelinked bundle of rank t . Given $r < t$, define the **prelinked Grassmannian** $\text{LG}(r, \mathcal{E}_\bullet)$ to be the scheme representing the functor

associating to an S -scheme T the set of all linked subbundles of rank r of the pullback of \mathcal{E}_\bullet to T .

Recall that linked subbundles were defined previously in Definition 4.3.

Thus, a point of a prelinked Grassmannian is itself an s -prelinked bundle, of rank r .

Definition 5.2. We say a point of $\mathrm{LG}(r, \mathcal{E}_\bullet)$ is **simple** (respectively, **internally simple** relative to m) if the induced s -prelinked bundle is simple (respectively, internally simple relative to m).

For our purposes, the main result we will need is the following:

Proposition 5.3. *On the locus of simple points, $\mathrm{LG}(r, \mathcal{E}_\bullet)$ is smooth over S of relative dimension $r(t - r)$.*

When $S = \mathrm{Spec} K$ is a point, at a particular K -valued simple point $(F_w)_{w \in V(G)}$, we may describe the tangent space as follows: given $W_w \subseteq F_w$ as in Proposition 2.7, the tangent space to $\mathrm{LG}(r, \mathcal{E}_\bullet)$ at $(F_w)_{w \in V(G)}$ is equal to

$$\bigoplus_{w \in V(G)} \mathrm{Hom}(W_w, \mathcal{E}_w / F_w).$$

See Proposition A.2.2 of [Oss14a].

We can now proceed to define linked alternating Grassmannians, and we easily conclude our main result on them.

Definition 5.4. Given \mathcal{E}_\bullet s -prelinked with a linked alternating form \langle, \rangle_\bullet , the **prelinked alternating Grassmannian** $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is the closed subscheme of $\mathrm{LG}(r, \mathcal{E}_\bullet)$ parametrizing linked subbundles which are isotropic for \langle, \rangle_\bullet .

Proof of Theorem 1.4. By definition, $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is precisely the isotropy locus of \langle, \rangle_\bullet for the universal subbundle on $\mathrm{LG}(r, \mathcal{E}_\bullet)$. Since the statement is local, we may restrict to the locus of simple points of $\mathrm{LG}(r, \mathcal{E}_\bullet)$. On this locus, the universal subbundle is simple, so Corollary 3.6 implies that $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is cut out locally by $\binom{r}{2}$ equations, as desired.

Next, recall that \langle, \rangle_\bullet induces a morphism from $\mathrm{LG}(r, \mathcal{E}_\bullet)$ to the space of linked alternating forms on the universal subbundle, and $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ is precisely the pullback of the zero section under this morphism. By Proposition 3.4, this space is (on the simple locus) a vector bundle \mathcal{G} of rank $\binom{r}{2}$. Letting s_0 be the zero section of \mathcal{G} and s_1 the section induced by \langle, \rangle_\bullet , we may view $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ as $s_0 \cap s_1$. In order to prove the theorem, it is then enough (see for instance Lemma 2.3 of [Oss13c]) to see that under the symplectic hypotheses, the fibers of $\mathrm{LAG}(r, \mathcal{E}_\bullet, \langle, \rangle_\bullet)$ are smooth of the correct dimension, or equivalently, that the tangent spaces to s_0 and s_1 intersect transversely in the fiber over any point of S . We may thus assume that S is a point, and consequently the \mathcal{E}_\bullet are simply vector spaces.

At any point of s_0 , the tangent space of \mathcal{G} decomposes canonically as a direct sum of the tangent space of $\mathrm{LG}(r, \mathcal{E}_\bullet)$ (which is described by Proposition 5.3) and the tangent space to the moduli space of linked alternating forms on the corresponding fixed linked subspace. Since the latter moduli space is a vector space, the tangent space is identified with the space itself. Then thinking of z as a point of $s_0 \cap s_1$, given a tangent vector v to $\mathrm{LG}(r, \mathcal{E}_\bullet)$ at a point, our tautological section s_1 induces a tangent vector of \mathcal{G} at z , and hence also a tangent vector v' of the fiber of \mathcal{G} at z . We may then think of v' as a linked alternating form. One checks from the

definitions that if the tangent vector v is represented by $(\varphi_i : \mathcal{F}_i \rightarrow \mathcal{E}_i/\mathcal{W}_i)_i$ for some choice of \mathcal{W}_i as in Proposition 2.7, the resulting linked alternating form obtained from v' is precisely the $\langle, \rangle_{\bullet}^{\varphi_{\bullet}}$ of Definition 4.4. Since tangent vectors to the zero section always yield the zero linked alternating form, transversality of the tangent spaces of the two sections follows from the surjectivity of the map $\varphi_{\bullet} \rightarrow \langle, \rangle_{\bullet}^{\varphi_{\bullet}}$, given to us by Lemma 4.5. We thus conclude the theorem. \square

6. SPECIAL LINE BUNDLES ON REDUCIBLE CURVES

In order to talk about higher-rank limit linear series with special determinant, we need to develop the appropriate definition of special line bundle on a curve of compact type. We make the following definition:

Definition 6.1. Let X be a curve of compact type over a field, and \mathcal{L} a line bundle on X . We say \mathcal{L} is **special** if there exists a morphism $\mathcal{L} \rightarrow \omega_X$ which is not everywhere zero on any component of X .

The definition is justified by the following proposition:

Proposition 6.2. Suppose that $\pi : X \rightarrow B$ is a smoothing family, with B the spectrum of a DVR, and with smooth generic fiber X_η . Let X_0 be the special fiber. Then:

- (i) If \mathcal{L}_η is a special line bundle on X_η , then it extends to a line bundle \mathcal{L} on X such that $\mathcal{L}|_{X_0}$ is also special.
- (ii) If \mathcal{L}_0 is a special line bundle on X_0 , then after possible faithfully flat base change $B' \rightarrow B$ with B' still the spectrum of a DVR, there is a line bundle \mathcal{L} on $X' = X \times_B B'$ and a morphism $\mathcal{L} \rightarrow \omega_{X'/B'}$ which does not vanish identically on any component of the special fiber X'_0 , and such that $\mathcal{L}|_{X'_0}$ is isomorphic to the pullback of \mathcal{L}_0 .

Recall the definition of smoothing family:

Definition 6.3. A morphism of schemes $\pi : X \rightarrow B$ constitutes a **smoothing family** if:

- (I) B is regular and connected;
- (II) π is flat and proper;
- (III) The fibers of π are genus- g curves of compact type;
- (IV) Each connected component Δ' of the singular locus of π maps isomorphically onto its scheme-theoretic image Δ in B , and furthermore $\pi^{-1}\Delta$ breaks into two (not necessarily irreducible) components intersecting along Δ' ;
- (V) Any point in the singular locus of π which is smoothed in the generic fiber is regular in the total space of X .

We first prove the following lemma.

Lemma 6.4. Let X_0 be a curve of compact type, and $w \in H_d$ a multidegree of total degree d , nonnegative on each component of X_0 . Then the space of \mathfrak{g}_w^0 s on X_0 has the expected dimension d .

Note that even though X_0 is reducible, we are considering linear series in the classical sense, and not limit linear series.

Proof. Let $G_w^0(X_0)$ be the space of \mathfrak{g}_w^0 s on X_0 , and denote by $G_w^{0,\text{nd}}(X_0)$ the open subset on which the global section is not identically zero on any component of X_0 . Let Γ be the dual graph of X_0 , so that w consists of a non-negative integer i_v for each $v \in V(\Gamma)$, and let Y_v denote the component of X_0 corresponding to a given v .

The argument giving the lower bound on dimension of $G_w^0(X_0)$ is unaffected by the reducibility of X_0 , so it suffices to prove that the dimension is at most d .

Now, we have a natural restriction map

$$(6.1) \quad G_w^{0,\text{nd}}(X_0) \rightarrow \prod_{v \in V(\Gamma)} G_{i_v}^0(Y_v),$$

and because $i_v \geq 0$ for each v , the target space has dimension $\sum_v i_v = d$. Because Γ is a tree, gluing conditions are independent, so the image consists of tuples $(\mathcal{L}_v, V_v)_{v \in V(\Gamma)}$ such that if V_v vanishes at a node corresponding to an edge e , and v' is the other vertex adjacent to e , then $V_{v'}$ likewise vanishes at the same node. Such vanishing also determines the fibers of (6.1): if sections vanish at a node, (6.1) fails to be injective due to the ability to scale independent on either side of the node. However, this is the only ambiguity, so we see that if we have $(\mathcal{L}_v, V_v)_{v \in V(\Gamma)}$ in the image of (6.1), and $S \subseteq E(\Gamma)$ is the set of nodes at which the V_v vanish, then the fiber of (6.1) over the given point has dimension $|S|$. On the other hand, the dimension of the locus in $\prod_{v \in V(\Gamma)} G_{i_v}^0(Y_v)$ with given S is $d - 2|S|$, so we conclude that $G_w^{0,\text{nd}}$ has dimension at most, hence equal to, d .

The argument on the complement of $G_w^{0,\text{nd}}$ is similar; let S' be a nonempty subset of $V(\Gamma)$, and $G_w^{0,S'}(X_0)$ the locally closed subscheme of $G_w^0(X_0)$ on which the sections vanish identically precisely on Y_v for $v \in S'$. Then we again have a restriction map

$$(6.2) \quad G_w^{0,S'}(X_0) \rightarrow \prod_{v \notin S'} G_{i_v}^0(Y_v),$$

and again using that every i_v is nonnegative, we have that the target space has dimension $d' := \sum_{v \notin S'} i_v \leq d$. In this case, the image consists of tuples $(\mathcal{L}_v, V_v)_{v \notin S'}$ such that if v is adjacent to a vertex in S' , then V_v vanishes at the relevant node, and if v, v' are adjacent and neither is in S' , then V_v vanishes at the relevant node if and only if $V_{v'}$ does. Fix a subset $S_1 \subseteq E(\Gamma)$ of edges such that neither adjacent vertex is in S' , and let S_2 consist of all edges such that exactly one adjacent vertex is in S' . Consider the locus of tuples $(\mathcal{L}_v, V_v)_{v \notin S'}$ which vanish precisely on the nodes corresponding to elements of $S := S_1 \cup S_2$. This locus has dimension $d' - 2|S_1| - |S_2|$. On the other hand, the fiber dimension of (6.2) is equal to $|S_1| + |S_2| - m$, where m is the number of connected components of $\cup_{v \in S'} Y_v$. Thus, we get that $\dim G_w^{0,S'}(X_0) < d' \leq d$, as desired. \square

Proof of Proposition 6.2. For (i), let D_η be the divisor on X_η giving the vanishing of a non-zero morphism $\mathcal{L}_\eta \rightarrow \omega_{X_\eta}$. Let D be the closure of D_η in X ; since X is assumed regular, this is a Cartier divisor, so we can set $\mathcal{L} = \omega_{X/B}(-D)$. This visibly has the desired properties.

For (ii), the morphism $\mathcal{L}_0 \rightarrow \omega_{X_0}$ can also be considered a global section of $\mathcal{L}_0^{-1} \otimes \omega_{X_0}$, not vanishing on any component of X_0 . Setting $d = 2g - 2 - \deg \mathcal{L}_0$, and w to be the multidegree of $\mathcal{L}^{-1} \otimes \omega_{X_0}$, by Lemma 6.4 the space of \mathfrak{g}_w^0 s on X_0 has expected dimension d , so our given one is the specialization of one from X_η . It follows that after base change $B' \rightarrow B$, we have a line bundle \mathcal{M} with a global

section not vanishing identically on any component of X'_0 , and such that $\mathcal{M}|_{X'_0}$ is the pullback of $\mathcal{L}_0^{-1} \otimes \omega_{X_0}$, so finally setting $\mathcal{L} = \mathcal{M}^{-1} \otimes \omega_{X'/B'}$ we obtain the desired statement. \square

7. LIMIT LINEAR SERIES WITH SPECIAL DETERMINANT

We now use Theorem 1.4 to prove Theorem 1.1. This is in essence a more detailed presentation of the sketch given in [OT14], although certain aspects are more complicated.

We assume throughout that X_0 is a genus- g curve of compact type with dual graph Γ over $\text{Spec } K$, with K an algebraically closed field, and denote by Y_v the component of X_0 corresponding to each $v \in V(\Gamma)$. We also fix integers d, k, b and d_\bullet as in Situation 2.1, and consequent associated graph G . We first recall the definitions involved in the statements of Theorem 1.1, starting with limit linear series as generalized in [Tei91a] from the construction of Eisenbud and Harris.

In order to facilitate precise statements of gluing conditions, we begin by introducing notation for twisting bundles used to go between different multidegrees.

Notation 7.1. For every vertex $v \in V(\Gamma)$, denote by \mathcal{O}_v the line bundle on X_0 obtained as follows: let Z_1, \dots, Z_n be the closures in X of the connected components of $X \setminus Y_v$, and for $i = 1, \dots, n$, let Δ'_i be the node of X_0 obtained as $Z_i \cap Y_v$. We then define \mathcal{O}_v to be $\mathcal{O}_{Y_v}(-\sum_i \Delta'_i)$ on Y_v and to be $\mathcal{O}_{Z_i}(\Delta'_i)$ on each Z_i .

Given $w, w' \in V(G)$, let P be a minimal directed path in G from w to w' . Let v_1, v_2, \dots, v_ℓ be the sequence in $V(\Gamma)$ induced by the edges making up P . Set

$$\mathcal{O}_{w,w'} = \bigotimes_{i=1}^{\ell} \mathcal{O}_{v_i}.$$

Now, given $v \in V(\Gamma)$, we have the stack $\mathcal{G}_{r,d_v}^k(Y_v)$ of \mathfrak{g}_{r,d_v}^k 's on Y_v . Given $e \in E(\Gamma)$ adjacent to v , we have also the stack $\mathcal{M}_r(\Delta'_e)$ of rank- r vector bundles on Δ'_e , where Δ'_e is the node of X_0 corresponding to the edge e . There is a natural restriction map $\mathcal{G}_{r,d_v}^k(Y_v) \rightarrow \mathcal{M}_r(\Delta'_e)$, but for reasons which will soon become apparent, we instead fix a choice of $w_0 \in V(G)$ and consider the map induced first by twisting the universal bundle by $(\mathcal{O}_{(w_0,w_0)}|_{Y_v})$, and then restricting to Δ'_e .

Notation 7.2. Denote by $\mathcal{P}_{r,d_\bullet}^k(X_0)$ the product of all the stacks $\mathcal{G}_{r,d_v}^k(Y_v)$ fibered over the stacks $\mathcal{M}_r(\Delta'_e)$ via the above maps.

Note that while $\mathcal{M}_r(\Delta'_e)$ has a single point, that point has automorphism group GL_r , so the stack is non-trivial. The purpose of fibering over it and twisting each vector bundle to multidegree w_0 is that, due to the definition of 2-fibered products, this process precisely introduces a choice of gluing map at each node. Thus, another way to express $\mathcal{P}_{r,d_\bullet}^k(X)$ is as a proper scheme over $\mathcal{M}_{r,w_0}(X_0)$ whose fibers parametrize tuples of spaces of global sections of the restriction of the given bundle to each component Y_v , twisted by $\mathcal{O}_{(w_0,w_0)}|_{Y_v}$ to obtain degree d_v . Here $\mathcal{M}_{r,w_0}(X_0)$ denotes the moduli stack of vector bundles of rank r and multidegree w_0 on X_0 .

We now recall the definition of Eisenbud-Harris-Teixidor limit linear series. To minimize confusion, we will use superscripts to index vector bundles on irreducible components, and subscripts to index bundles on the entire curve. For convenience in gluing map notation, we choose directions for all edges of Γ .

Definition 7.3. Let $((\mathcal{E}^v, V^v)_{v \in V(\Gamma)}, (\varphi_e)_{e \in E(\Gamma)})$ be a K -valued point of $\mathcal{P}_{r,d_\bullet}^k(X)$, where (\mathcal{E}^v, V^v) is the corresponding point of $\mathcal{G}_{r,d_v}^k(Y_v)$, and if e is an edge from v to v' ,

$$\varphi_e : (\mathcal{E}^v \otimes (\mathcal{O}_{(w_v, w_0)}|_{Y_v}))|_{\Delta'_e} \xrightarrow{\sim} (\mathcal{E}^{v'} \otimes (\mathcal{O}_{(w_{v'}, w_0)}|_{Y_{v'}}))|_{\Delta'_e}$$

is the corresponding gluing isomorphism. Then $((\mathcal{E}_v, V_v)_v, (\varphi_e)_e)$ is an **Eisenbud-Harris-Teixidor limit linear series** if for each $e \in E(\Gamma)$ adjacent to v and v' , we have:

- (I) $H^0(Y_v, \mathcal{E}_v(-(b+1)\Delta'_e)) = 0$ and similarly for v' ;
- (II) if $a_1^{e,v}, \dots, a_k^{e,v}$ and $a_1^{e,v'}, \dots, a_k^{e,v'}$ are the vanishing sequences at Δ'_e for (\mathcal{E}_v, V_v) and $(\mathcal{E}_{v'}, V_{v'})$ respectively, then for every i we have

$$(7.1) \quad a_i^{e,v} + a_{k+1-i}^{e,v'} \geq b;$$

- (III) there exist bases $s_1^{e,v}, \dots, s_k^{e,v}$ and $s_1^{e,v'}, \dots, s_k^{e,v'}$ of V_v and $V_{v'}$ respectively such that $s_i^{e,v}$ has vanishing order $a_i^{e,v}$ at Δ'_e for each i , and similarly for $s_i^{e,v'}$, and if we have $a_i^{e,v} + a_{k+1-i}^{e,v'} = b$ for some i , then

$$(7.2) \quad \varphi_e(s_i^{e,v}) = s_{k+1-i}^{e,v'};$$

We say that $((\mathcal{E}_v, V_v)_v, (\varphi_e)_e)$ is **refined** if equality always holds in (7.1).

In imposing (7.2), we view each section in the appropriate twist determined by its order of vanishing; see Remark 4.1.4 of [Oss14a] for details. *A priori*, the Eisenbud-Harris-Teixidor limit linear series only form a set, but in Definition 4.2.1 of [Oss14a] we endow it with a natural stack structure.

Notation 7.4. We denote by $\mathcal{G}_{r,d_\bullet}^{k,\text{EHT}}(X_0)$ the stack of Eisenbud-Harris-Teixidor limit linear series.

A limit linear series of fixed determinant is essentially as one would expect, with the only subtlety being that twisting is allowed after taking the determinant. This eliminates parity restrictions on degrees, which turn out to be unnecessary.

Definition 7.5. Given a line bundle \mathcal{L} on X_0 of multidegree $w \in H_d$, the stack $\mathcal{G}_{r,\mathcal{L},d_\bullet}^{k,\text{EHT}}(X_0)$ of **Eisenbud-Harris-Teixidor limit linear series of determinant \mathcal{L}** is the (2-)fibered product of $\mathcal{G}_{r,d_\bullet}^{k,\text{EHT}}(X_0)$ with $\text{Spec } K$ over the Picard stack $\mathcal{P}ic^w(X_0)$, with the map $\mathcal{G}_{r,d_\bullet}^{k,\text{EHT}}(X_0) \rightarrow \mathcal{P}ic^w(X_0)$ given by taking the determinant of the underlying bundle and twisting appropriately, and the map $\text{Spec } K \rightarrow \mathcal{P}ic^w(X_0)$ obtained by considering \mathcal{L} as a point on the Picard stack.

Thus, if we write $\mathcal{M}_{r,\mathcal{L}}(X_0)$ for the moduli stack of vector bundles of rank r and determinant \mathcal{L} on X_0 , there is a forgetful morphism $\mathcal{G}_{r,\mathcal{L},d_\bullet}^{k,\text{EHT}}(X_0) \rightarrow \mathcal{M}_{r,\mathcal{L}}(X_0)$.

To make the twist in Definition 7.5 more precise, we introduce the following notation, which is essentially the rank-1 analogue of Notation 7.1:

Notation 7.6. Given $w, w' \in H_d$, let

$$\mathcal{O}'_{w,w'} = \bigotimes_i \mathcal{O}_{v_i},$$

where $v_1, \dots, v_n \in V(\Gamma)$ is a minimal sequence such that for each $v \in V(\Gamma)$, the v th entry of $w' - w$ is given by

$$\#\{i : v_i \text{ is adjacent to } v\} - \ell \cdot \#\{i : v_i = v\},$$

where ℓ is the valence of v .

Then the twist used in Definition 7.5 after taking the determinant consists of tensoring with $\mathcal{O}'_{w_0, w}$.

Thus, the determinant \mathcal{L} condition for a given limit linear series requires that we fix an isomorphism between \mathcal{L} and the appropriate twist of the determinant of an underlying vector bundle. In particular, this includes extra data which has the effect of rigidifying the stack slightly, and increasing the dimension by 1. We denote the isomorphism $\det \mathcal{E} \otimes \mathcal{O}'_{w_0, w} \xrightarrow{\sim} \mathcal{L}$ by ψ . Although the particular choice of ψ will never be relevant, having made a choice will affect dimension counts in the stack context.

We also recall the definition of chain-adaptability.

Definition 7.7. Let X be a smooth projective curve over $\text{Spec } F$, and (\mathcal{E}, V) a pair with \mathcal{E} a vector bundle of rank r on X , and V a k -dimensional space of global sections. Given points $P, Q \in X(F)$, let $a_1(P), \dots, a_k(P)$ and $a_1(Q), \dots, a_k(Q)$ be the vanishing sequences at P and Q respectively. Then we say that a basis $s_1, \dots, s_k \in V$ is (P, Q) -**adapted** if $\text{ord}_P s_i = a_i(P)$ and $\text{ord}_Q s_i = a_{k+1-i}(Q)$ for $i = 1, \dots, k$. We say that (\mathcal{E}, V) is (P, Q) -**adaptable** if there exists a (P, Q) -adapted basis of V .

Definition 7.8. Let X_0 be a curve consisting of a chain of smooth projective curves X_1, \dots, X_n over $\text{Spec } F$, with $P_i, Q_i \in X_i(F)$ for each i , and the point Q_i on X_i glued to P_{i+1} on X_{i+1} . Then a refined Eisenbud-Harris-Teixidor limit linear series on X_0 is **chain adaptable** if the pair induced by restriction to each X_i ($i = 2, \dots, n-1$) is (P_{i-1}, P_i) -adaptable.

Now all of the concepts involved in Theorem 1.1 have been defined. The reason for stating the theorem in terms of the above-defined limit linear series is that they are comparatively tractable in practice. However, they do not lend themselves easily to theoretic arguments, such as the desired smoothing theorem. In contrast, we now recall the notion of linked linear series, which are not effective computational tools, but are more amenable to theoretical constructions. In the language of [Oss14a], we will be using “type II linked linear series,” the type I variant not being relevant for our purposes. In fact, we will be using a slight variant which is shown to be equivalent in Proposition 3.4.12 of [Oss14a].

Because we wish to prove smoothing theorems, we have to work in families. The only change to our running hypotheses is that in place of X_0 , we consider a smoothing family $\pi : X \rightarrow B$, and in addition to the case that the base B is a point as considered above, we also allow B to be the spectrum of a DVR, with $\pi : X \rightarrow B$ a smoothing family having smooth generic fiber. In this case, Γ will denote the dual graph of the special fiber, and Y_v is the appropriate component of the special fiber. Accordingly, we have to generalize our definition of twisting bundles.

Notation 7.9. In the case that B is the spectrum of a DVR, we have Y_v a Cartier divisor in X , and we set $\mathcal{O}_v = \mathcal{O}_X(Y_v)$.

We then define $\mathcal{O}_{w, w'}$ as before.

Finally, for any B , given $w_0 \in V(G)$, and a T -valued point \mathcal{E} of $\mathcal{M}_{r, w_0}(X/B)$, for $w \in V(G)$ write

$$\mathcal{E}_w := \mathcal{E} \otimes \mathcal{O}_{w_0, w}|_{X \times_B T}.$$

We will need to choose some additional data to define linked linear series, which we use to construct maps as described below.

Notation 7.10. For each vertex $v \in V(\Gamma)$, suppose we fix a morphism

$$\iota_v : \mathcal{O}_X \rightarrow \mathcal{O}_v$$

vanishing precisely on Y_v .

Next, observe that $\bigotimes_{v \in V(\Gamma)} \mathcal{O}_v \cong \mathcal{O}_X$. Fixing such an isomorphism (unique up to an element of \mathcal{O}_B^*), we obtain an induced morphism

$$\iota'_v : \bigotimes_{v' \neq v} \mathcal{O}_{v'} \xrightarrow{\iota_v} \bigotimes_{v' \in V(\Gamma)} \mathcal{O}_{v'} \rightarrow \mathcal{O}_X.$$

Finally, fix $w_0 \in V(G)$. For any edge ε in G , let w be the tail and w' the head, and let v be the associated edge of Γ . If \mathcal{E} is a T -valued point of $\mathcal{M}_{r,w_0}(X/B)$ then we either have $\mathcal{E}_{w'} = \mathcal{E}_w \otimes \mathcal{O}_v$ or $\mathcal{E}_w = \mathcal{E}_{w'} \otimes \bigotimes_{v' \neq v} \mathcal{O}_{v'}$. Thus, using ι_v or ι'_v , and pushing forward under π , we obtain a morphism

$$f_\varepsilon : \pi_* \mathcal{E}_w \rightarrow \pi_* \mathcal{E}_{w'}.$$

Definition 7.11. Choose a vertex $w_0 \in V(G)$. In addition, for each $v \in V(\Gamma)$, choose a morphism $\iota_v : \mathcal{O}_X \rightarrow \mathcal{O}_v$ which vanishes precisely on Y_v . The moduli stack $\mathcal{G}_{r,w_0,d_\bullet}^k(X/B)$ of **linked linear series** is the category fibered in groupoids over B -Sch whose objects consist of tuples $(S, \mathcal{E}, (\mathcal{V}_w)_{w \in V(G)})$, where S is a B -scheme, \mathcal{E} is a vector bundle of rank r and multidegree w_0 on $X \times_B S$, and \mathcal{V}_w is a rank- k subbundle (in the sense of Definition B.2.1 of [Oss14a]) of $\pi_*(\mathcal{E}_w)$, satisfying the following conditions:

- (I) for every $v \in V(\Gamma)$, every $e \in E(\Gamma)$ adjacent to v , and every $z \in S$ mapping to the closed point of B , we have

$$H^0(Y, \mathcal{E}_{w_v}|_Y(-(b+1)(\Delta')) = 0,$$

where Y is the component of the fiber X_z corresponding to v , and Δ' denotes the node corresponding to e .

- (II) For every edge ε in G , let w be the tail and w' the head. Then we require that

$$f_\varepsilon(\mathcal{V}_w) \subseteq \mathcal{V}_{w'}.$$

Given $B' \rightarrow B$ and a line bundle \mathcal{L} on $X_{B'}$ of multidegree $w \in H_d$, the moduli stack $\mathcal{G}_{r,\mathcal{L},d_\bullet}^k(X/B)$ of **linked linear series of determinant \mathcal{L}** is the (2-)fibered product of $\mathcal{G}_{r,w_0,d_\bullet}^k(X/B)$ with B' over $\mathcal{P}ic^w(X/B)$, as in the case of limit linear series.

In [Oss14a], linked linear series are shown to be parametrized by algebraic stacks. It is always possible to increase b and the d_v , in which case one gets an open immersion into a larger moduli stack; see Proposition 6.2.1 of [Oss14a]. Although in general the stacks of limit linear series and linked linear series are rather different, the importance of chain-adaptability from our point of view is that it describes an open substack on which the two constructions are isomorphic.

Theorem 7.12. *Suppose B is a point, and write $X_0 = X$. Then there is a forgetful morphism*

$$(7.3) \quad \mathcal{G}_{r,w_0,d_\bullet}^k(X) \rightarrow \mathcal{G}_{r,d_\bullet}^{k,\text{EHT}}(X)$$

which is an isomorphism over the open substack of chain-adaptable limit linear series.

See Theorem 4.3.4 and Corollary 5.2.7 of [Oss14a].

Thus, in order to prove Theorem 1.1, we may restrict our attention to linked linear series. A slight specialization of Theorem 6.1.4 of [Oss14a] says the following.

Theorem 7.13. *Let $\pi : X \rightarrow B$ be a smoothing family with B being the spectrum of a DVR, and having smooth generic fiber X_η , and special fiber X_0 . Given also k, r, d_\bullet as in Situation 2.1, let \mathcal{L} be a line bundle of multidegree $w \in H_d$ on $X \times_B B'$ for some B -scheme B' . Then $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B)$ is an Artin stack over B' , and the natural map*

$$\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B) \rightarrow \mathcal{M}_{r, \mathcal{L}}(X_{B'}/B')$$

is relatively representable by schemes of finite type. Moreover, formation of the stack $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B)$ is compatible with base change $B'' \rightarrow B$, and in particular, if η' is a point of B' over the generic point of B , the base change to η' parametrizes triples $(\mathcal{E}, \mathcal{L}, V)$ of a vector bundle \mathcal{E} of rank r and degree d on $X_{\eta'}$ together with an isomorphism $\det \mathcal{E} \xrightarrow{\sim} \mathcal{L}|_{\eta'}$ and a k -dimensional vector space $V \subseteq H^0(X_{\eta'}, \mathcal{E})$.

Moreover, the simple locus of $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B)$ has universal relative dimension at least $k(d - k - r(g - 1))$ over $\mathcal{M}_{r, \mathcal{L}}(X'/B')$, and therefore universal relative dimension at least $\rho - g$ over B' . In particular, if $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X_0)$ has dimension exactly $\rho - g$ at a simple point z , then $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B)$ is universally open over B at z , and has fibers of pure dimension $\rho - g$ in an open neighborhood of z .

The notion of universal relative dimension is introduced in [Oss13b].

The above theorem is valid for any determinant \mathcal{L} , but the last part is not useful when \mathcal{L} is special, since the fiber will always have dimension at least $\rho - g + \binom{k}{2}$. Thus, our main task is to prove the following variant which addresses the case of special determinants.

Theorem 7.14. *In the situation of Theorem 7.13, suppose further that \mathcal{L} is special.*

Then the simple locus of $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X/B)$ has universal relative dimension at least $\rho - g + \binom{k}{2}$ over B . In particular, if $\mathcal{G}_{r, \mathcal{L}, d_\bullet}^k(X_0)$ has dimension exactly $\rho - g + \binom{k}{2}$ at a simple point z , then the structure morphism is universally open in a neighborhood of z , with pure fiber dimension $\rho - g + \binom{k}{2}$.

In families, we say \mathcal{L} is special if there is a morphism $\mathcal{L} \rightarrow \omega_{X/B}$ which does not vanish uniformly on any component of any fiber.

Before describing the limit case, we briefly recall a derivation for smooth curves (see §2 of [BF98] or Theorem 4.2 of [Muk95]) of the modified expected dimension in the special case of rank 2 and canonical determinant, via the following alternate construction of the moduli space in question. Let $\mathcal{M}_{2, \omega}(X)$ be the moduli stack of vector bundles of rank 2 and fixed canonical determinant on a smooth curve X of genus g ; this is smooth of dimension $3g - 3$. Let $\tilde{\mathcal{E}}$ be the universal bundle on $\mathcal{M}_{2, \omega}(X) \times X$, and let D be a sufficiently ample effective divisor on X (technically, we must cover $\mathcal{M}_{2, \omega}(X)$ by a nested increasing sequence of quasicompact open substacks, and carry out this construction on each, letting D grow). Let D' be the pullback of D to $\mathcal{M}_{2, \omega}(X) \times X$. Then $p_{1*} \tilde{\mathcal{E}}(D')$ is a vector bundle of rank

$$\deg \tilde{\mathcal{E}} + \operatorname{rk} \tilde{\mathcal{E}} \deg D + \operatorname{rk} \tilde{\mathcal{E}}(1 - g) = 2g - 2 + 2 \deg D + 2 - 2g = 2 \deg D.$$

Let $G := G(k, p_{1*}\tilde{\mathcal{E}}(D'))$ be the relative Grassmannian on $\mathcal{M}_{2,\omega}(X)$; our moduli space $\mathcal{G}_{2,\omega}^k(X)$ is cut out by the closed condition of subspaces lying in $p_{1*}\tilde{\mathcal{E}}$. We express this condition in terms of the bundle $p_{1*}(\tilde{\mathcal{E}}(D')/\tilde{\mathcal{E}}(-D'))$, which has rank $4 \deg D$. We see that because D was chosen to be large, $p_{1*}\tilde{\mathcal{E}}(D')$ is naturally a subbundle, as is $p_{1*}(\tilde{\mathcal{E}}/\tilde{\mathcal{E}}(-D'))$, which also has rank $2 \deg D$. Then the inclusion of the universal subbundle on G , together with the pullback from $\mathcal{M}_{2,\omega}(X)$ of $p_{1*}(\tilde{\mathcal{E}}/\tilde{\mathcal{E}}(-D'))$, induces a morphism

$$(7.4) \quad G \rightarrow G(k, p_{1*}(\tilde{\mathcal{E}}(D')/\tilde{\mathcal{E}}(-D'))) \times_{\mathcal{M}_{2,\omega}(X)} G(2 \deg D, p_{1*}(\tilde{\mathcal{E}}(D')/\tilde{\mathcal{E}}(-D'))),$$

and our desired moduli space is precisely the preimage in G of the incidence correspondence in the product. The incidence correspondence is smooth inside a smooth product space over $\mathcal{M}_{2,\omega}(X)$, so is a local complete intersection, and we thus obtain an upper bound on the codimension of $\mathcal{G}_{2,\omega}^k(X)$ inside G .

However, this lower bound is not sharp: we next make use of the canonical determinant hypothesis to observe that by choosing local representatives, using the isomorphism $\bigwedge^2 \tilde{\mathcal{E}} \cong p_2^* \omega_X$, and summing residues over points of D , we obtain a symplectic form on $p_{1*}(\tilde{\mathcal{E}}(D')/\tilde{\mathcal{E}}(-D'))$. Moreover, both $p_{1*}\tilde{\mathcal{E}}(D')$ and $p_{1*}(\tilde{\mathcal{E}}/\tilde{\mathcal{E}}(-D'))$ are isotropic for this form, with the former following from the residue theorem, and the latter from the lack of poles. Thus, (7.4) in fact has its image in a product of symplectic Grassmannians, and the incidence correspondence has smaller codimension, and is still a local complete intersection, so we obtain the modified codimension bound for $\mathcal{G}_{2,\omega}^k(X)$ cut out in G . In fact, in the language of [Oss13b], we obtain that the moduli space has universal relative dimension at least $\rho - g + \binom{k}{2}$ from Corollary 7.7 of [Oss13b].

We now prove the corresponding dimension bounds for limit linear series, by following the above argument and applying Theorem 1.4.

Proof of Theorem 7.14. First, Proposition 6.2.1 of [Oss14a] shows that suitable choices for increasing d_\bullet and b will always induce an open immersion of the corresponding spaces $\mathcal{G}_{r,\mathcal{L},d_\bullet}^k(X/B)$, so to prove a local dimension bound we may increase d_\bullet and b as much as we wish. In particular, if z is a simple point of $\mathcal{G}_{r,\mathcal{L},d_\bullet}^k(X/B)$, then we can increase d_\bullet and b so that z becomes internally simple. Now, let $\tilde{\mathcal{E}}_0$ be the universal bundle on $\mathcal{M}_{2,\mathcal{L}}(X'/B')$, and for each $w \in V(G)$, let $\tilde{\mathcal{E}}_w$ be the corresponding twist of $\tilde{\mathcal{E}}_0$. Then for every edge ε of G , we also have an induced morphism $\tilde{f}_\varepsilon : \tilde{\mathcal{E}}_{t(\varepsilon)} \rightarrow \tilde{\mathcal{E}}_{h(\varepsilon)}$. Let D and D' be as in the smooth case described above, with the added condition that D be supported on the smooth locus of π . Then we can consider for each $w \in V(G)$ the vector bundle $p_{1*}(\tilde{\mathcal{E}}_w(D')/\tilde{\mathcal{E}}_w(-D'))$, with morphisms f_ε induced by \tilde{f}_ε . This creates an s -prelinked structure $\tilde{\mathcal{G}}_\bullet$, and the relative moduli space $\mathcal{G}_{r,\mathcal{L},d_\bullet}^k(X/B)$ of linked linear series may be realized inside the linked Grassmannian $\text{LG}(k, \tilde{\mathcal{G}}_\bullet)$. As in the smooth case, $\mathcal{G}_{r,\mathcal{L},d_\bullet}^k(X/B)$ is described as the locus on which the universal subbundle is contained inside both $p_{1*}(\tilde{\mathcal{E}}_w(D'))$ and $p_{1*}(\tilde{\mathcal{E}}_w/\tilde{\mathcal{E}}_w(-D'))$ for each w , so we are in exactly the situation of (7.4), with linked Grassmannians in place of Grassmannians.

We thus need to see that the special determinant hypothesis gives us (at least locally on $\mathcal{M}_{2,\mathcal{L}}(X'/B')$) a linked alternating form on $\tilde{\mathcal{G}}_\bullet$, which is internally symplectic with respect to the multidegree m of \mathcal{L} , considered in H_d , and such that

the subbundles in question are isotropic for this form. Now, by definition, we have maps

$$\mathcal{E}_{w_0} \otimes \mathcal{E}_{w_0} \otimes \mathcal{O}'_{w_0, m} \rightarrow \left(\bigwedge^2 \mathcal{E}_{w_0} \right) \otimes \mathcal{O}'_{w_0, m} \xrightarrow{\sim} \mathcal{L} \rightarrow \omega_{X'/B'}.$$

Given $w, w' \in V(G)$, the line bundle $\mathcal{O}_{w, w_0} \otimes \mathcal{O}_{w', w_0} \otimes \mathcal{O}'_{w_0, m}$ is of the form $\bigotimes_i \mathcal{O}_{v_i}$ for a sequence of $v_i \in V(\Gamma)$, and if we let v'_j be the sequence obtained from the v_i by removing all copies of $V(\Gamma)$, and set $\mathcal{O}_{w, w', m} = \bigotimes_j \mathcal{O}_{v'_j}$, then we have

$$\mathcal{O}_{w, w', m} \cong \mathcal{O}_{w, w_0} \otimes \mathcal{O}_{w', w_0} \otimes \mathcal{O}'_{w_0, m},$$

so we get induced morphisms

$$\begin{aligned} \mathcal{E}_w \otimes \mathcal{E}_{w'} &= \mathcal{E}_{w_0} \otimes \mathcal{O}_{w_0, w} \otimes \mathcal{E}_{w_0} \otimes \mathcal{O}_{w_0, w'} \rightarrow \mathcal{E}_{w_0} \otimes \mathcal{O}_{w_0, w} \otimes \mathcal{E}_{w_0} \otimes \mathcal{O}_{w_0, w'} \otimes \mathcal{O}'_{\frac{w+w'}{2}, m} \\ &\xrightarrow{\sim} \mathcal{E}_{w_0} \otimes \mathcal{E}_{w_0} \otimes \mathcal{O}'_{w_0, m} \rightarrow \mathcal{L} \rightarrow \omega_{X'/B'}, \end{aligned}$$

which if $w = w'$ will factor through $\bigwedge^2 \mathcal{E}_w$.

Applying the same method of choosing representatives and summing over residues as described in the smooth case above, we obtain a linked alternating form on $\tilde{\mathcal{G}}_\bullet$. Now, if $w + w' = 2m$, we have that $\mathcal{O}'_{\frac{w+w'}{2}, m}$ is trivial, so one checks by a local calculation that our linked alternating form is in fact internally symplectic. As in the smooth case, it is clear that pairing sections of $p_{1*}(\tilde{\mathcal{E}}_w / \tilde{\mathcal{E}}_w(-D'))$ with $p_{1*}(\tilde{\mathcal{E}}_{w'} / \tilde{\mathcal{E}}_{w'}(-D'))$ does not give any poles for the associated differentials, so all residues vanish, while pairing $p_{1*}(\tilde{\mathcal{E}}_w(D'))$ with $p_{1*}(\tilde{\mathcal{E}}_{w'}(D'))$ gives a global section of $\omega_{X'/B'}(2D)$, and again the residue theorem implies that summing over residues gives zero. Thus, both of these linked subbundles are isotropic.

We can thus put together the canonical determinant construction for the smooth case with the linked linear series construction simply by replacing the symplectic Grassmannians arising in the smooth case with prelinked alternating Grassmannians. Using that our form is internally symplectic, Theorem 1.4 tells us that these spaces are smooth of dimension equal to the usual symplectic Grassmannian (at least, on the open locus of internally simple points), so the dimension count in the linked linear series case goes through exactly as in the case of smooth curves, and we obtain the desired lower bound on universal relative dimension. The universal openness assertion then follows from Corollary 7.4 of [Oss13b]. \square

We conclude by describing how our main theorem follows from Theorem 7.14. Recall that rather than using standard stability for reducible curves, we will find it more convenient to use the notion of ℓ -stability introduced in [Oss14b]. For the sake of propriety, we give the relevant definition prior to the proof, although it will not be used.

Definition 7.15. Let \mathcal{E} be a vector bundle of rank r on a nodal X . We say that \mathcal{E} is **ℓ -semistable** (respectively, **ℓ -stable**) if for all proper subsheaves $\mathcal{F} \subseteq \mathcal{E}$ having constant rank r' on every component of X , we have

$$(7.5) \quad \frac{\chi(\mathcal{F})}{r'} \leq \frac{\chi(\mathcal{E})}{r} \quad \left(\text{respectively, } \frac{\chi(\mathcal{F})}{r'} < \frac{\chi(\mathcal{E})}{r} \right).$$

Note that ℓ -(semi)stability is trivially equivalent to usual (semi)stability in the case that X is smooth.

Definition 7.16. A limit linear series $((\mathcal{E}^v, V^v)_v, (\varphi_e)_e)$ of degree d on X is ℓ -**semistable** (respectively, ℓ -**stable**) if the underlying vector bundle \mathcal{E} of multidegree w_0 on X is ℓ -semistable (respectively, ℓ -stable).

Proof of Theorem 1.1. With X_0 as in the theorem statement, let $\pi : X \rightarrow B$ be a smoothing family with B the spectrum of a DVR, with special fiber X_0 , and with the generic fiber X_η being smooth. According to Theorem 7.12, the hypotheses of the theorem imply that the simple locus of $\mathcal{G}_{2, d_\bullet, \mathcal{L}_0}^k(X_0)$ has at least one component of dimension equal to $\rho - g + \binom{k}{2}$. By Proposition 6.2, we have that after possible faithfully flat base change $B' \rightarrow B$ with B' still the spectrum of a DVR, \mathcal{L}_0 is the restriction of some \mathcal{L} on $X' := X \times_B B'$, with $\mathcal{L}_\eta := \mathcal{L}_{X'_\eta}$ also special. Then by Theorem 7.14 it follows that $\mathcal{G}_{2, \mathcal{L}_\eta}^k(X'_\eta)$ has at least one component of dimension equal to $\rho - g + \binom{k}{2}$. Since $k \geq 2$, by Theorem 1.3 of [Oss13a], we conclude that $h^1(X'_\eta, \mathcal{L}_\eta)$ must be equal to 1. In particular, we must have $d \geq g - 2$, so that on a general curve a general special line bundle of degree d has h^1 equal to 1.

We have thus produced a single smooth curve X_η with line bundle \mathcal{L}_η such that $h^1(X_\eta, \mathcal{L}_\eta) = 1$ and $\mathcal{G}_{2, \mathcal{L}_\eta}^k(X'_\eta)$ has at least one component of dimension equal to $\rho - g + \binom{k}{2}$. To conclude the statement of the theorem, it is enough to apply the dimensional lower bounds of Corollary 1.2 of [Oss13a] to a versal family in which X_η varies over an open subset of curves of genus g , and \mathcal{L}_η varies over an open subset of special line bundles. Although *loc. cit.* is not stated for families, the argument goes through unchanged in this context; see also Theorem 3.4 of [Oss13c], where essentially the same argument is developed in families. Note that here we also use that the locus of special line bundles is irreducible on a general curve: for $d = 2g - 2$, this is because the canonical bundle is the unique choice, while for $d < 2g - 2$, this is a consequence of classical Brill-Noether theory (see [FL81] and [Gie82]).

Finally, the assertions on (semi)stability follow from openness in families, which is Proposition 6.4.4 of [Oss14a]. \square

8. SPECIFIC EXISTENCE RESULTS

We now explicitly produce families of limit linear series having special determinant and (the appropriately modified) expected dimension, allowing us to apply Theorem 1.1 to conclude Theorem 1.2.

We will work with the following situation throughout this section.

Situation 8.1. Let X_1, \dots, X_g be genus-1 curves, with marked points P_i, Q_i such that $P_i - Q_i$ is not m -torsion for any m with $1 \leq m \leq 2d$. Let X be obtained by gluing Q_i to P_{i+1} for $i = 1, \dots, g-1$. Fix d with $g-2 \leq d \leq 2g-2$. For $i = 1, \dots, g$ set

$$D_i = \begin{cases} Q_1 : & i = 1 \\ Q_i + P_i : & 1 < i < g \\ P_g : & i = g. \end{cases}$$

In this situation, we write $((\mathcal{E}^i, V^i)_{i=1, \dots, g}, (\varphi_j)_{j=1, \dots, g-1})$ to denote a limit linear series on X , where (\mathcal{E}^i, V^i) is on the component X_i , and φ_j is a gluing map from X_j to X_{j+1} .

Recall that the canonical line bundle \mathcal{L} on X is determined by $\mathcal{L}|_{X_i} = \mathcal{O}_{X_i}(D_i)$ for each i .

Proposition 8.2. *Let \mathcal{L} be a line bundle on C of degree d such that if $i \leq d+2-g$ we have $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i}(D_i)$, and for $i > d+2-g$ we have $\deg \mathcal{L}|_{X_i} = \deg D_i - 1$, and $\mathcal{L}|_{X_i} \not\cong \mathcal{O}_{X_i}(D)$ for $0 \leq D < D_i$. Then \mathcal{L} is special.*

Moreover, the degree- d aspect \mathcal{L}_i of \mathcal{L} on X_i for $1 \leq i \leq d+2-g$ is equal to $\mathcal{O}_{X_i}(2(i-1)P_i + (d-2i+2)Q_i)$. If we fix $d_\bullet = (d-1, \dots, d-1, d, d-1, \dots, d-1)$, with $d_j = d$, then the degree- d_i aspect of \mathcal{L}_i of \mathcal{L} on X_i for $1 \leq i \leq d+2-g$ is given by

$$\mathcal{L}_i \cong \begin{cases} \mathcal{O}_{X_i}(2(i-1)P_i + (d-2i+1)Q_i) : & i < j \\ \mathcal{O}_{X_i}(2(i-1)P_i + (d-2i+2)Q_i) : & i = j \\ \mathcal{O}_{X_i}((2i-3)P_i + (d-2i+2)Q_i) : & i > j. \end{cases}$$

In the above, the “degree- d_i aspect” is the line bundle on X_i obtained from \mathcal{L} as follows: if w is the multidegree of \mathcal{L} , b is determined by $\sum_i d_i - (g-1)b = d$, and w_i is given by d_i in index i and $d_j - b$ for all $j \neq i$, then the degree- d_i aspect of \mathcal{L} is the restriction to X_i of $\mathcal{L} \otimes \mathcal{O}'_{w, w_i}$.

Proof. Given the description of the canonical line bundle on X , the speciality is immediate from the definitions. Next, let c_i be the degree of $\mathcal{L}|_{X_i}$ for each i ; then we have $c_i = 2 - \delta_i - \epsilon_i$, where $\delta_i = \begin{cases} 1 : & i = 1, g \\ 0 : & \text{otherwise,} \end{cases}$ and $\epsilon_i = \begin{cases} 0 : & i \leq d+2-g \\ 1 : & \text{otherwise.} \end{cases}$

Then for the degree- d aspects, we have

$$\mathcal{L}_i = \mathcal{L}|_{X_i}((c_1 + \dots + c_{i-1})P_i + (c_{i+1} + \dots + c_g)Q_i),$$

and it is enough to verify that $c_1 + \dots + c_{i-1} + 1 = 2(i-1)$ for $1 < i \leq d+2-g$, which is clear. The degree- d_i aspects for the variant multidegree are computed similarly. \square

In this situation, we can give very explicit criteria for ℓ -(semi)stability as follows.

Proposition 8.3. *Given a limit linear series $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j)$ of degree d on X , suppose that each \mathcal{E}^i is semistable. Then the limit linear series is ℓ -semistable. If further there do not exist subbundles $\mathcal{F}^i \subseteq \mathcal{E}^i$ of equal rank $r' < r$ such that $\chi(\mathcal{F}^i)/r' = \chi(\mathcal{E}^i)/r$ for all i , and each \mathcal{F}^i glues to \mathcal{F}^{i+1} under φ_i , then the limit linear series is ℓ -stable.*

Proof. Corollary 1.7 of [Oss14b] gives the analogous statements for a single vector bundle on X and its restrictions to the X_i . Because semistability on components and the gluing maps between components are both preserved under twists, the desired statement follows. \square

The following notation will be convenient.

Notation 8.4. If $V \subseteq \Gamma(X', \mathcal{E})$ for a vector bundle \mathcal{E} on a smooth projective curve X' , and D is an effective divisor on X' , set

$$V(-D) := V \cap \Gamma(X', \mathcal{E}(-D)).$$

As a final preparatory step, we state the following result, which is important for analyzing possibilities for limit linear series with prescribed vanishing sequences, and which is an immediate consequence of Proposition 5.1.3 of [Oss14a].

Proposition 8.5. *Given a pair (\mathcal{E}, V) of a vector bundle on a smooth projective curve X' together with a k -dimensional space V of global sections of \mathcal{E} , and given also distinct points $P, Q \in X'$, suppose that the vanishing sequence of (\mathcal{E}, V) at P (respectively, Q) is given by a_1, \dots, a_k (respectively, b_1, \dots, b_k). Then given nonnegative integers a, b , we have*

$$\dim V(-aP - bQ) \geq \#\{i : a_i \geq a \text{ and } b_{k+1-i} \geq b\}.$$

In particular, if we have

$$\dim \Gamma(X', \mathcal{E}(-aP - bQ)) = \#\{i : a_i \geq a \text{ and } b_{k+1-i} \geq b\},$$

then $\Gamma(X', \mathcal{E}(-aP - bQ)) \subseteq V$.

Additionally, for any i we must have a section in V vanishing to order at least a_i at P and at least b_{k+1-i} at Q .

We now proceed to the limit linear series arguments which lead to Theorem 1.2. We have four cases to consider, according to the parity of d and of k (thus, only the first two cases were necessary to consider in the case of canonical determinant). Because the families described are somewhat complicated, we have included a small example of each family in Appendix A.

Proposition 8.6. *Assume that $d = 2d'$ and $k = 2k'$ are even, and set $d_i = d$ for $i = 1, \dots, g$. Then the b of Situation 2.1 is determined to be d' . Fix \mathcal{L} as in Proposition 8.2, and assume further that for $i > d + 2 - g$, we have $\mathcal{L}_i \not\cong \mathcal{O}_{X_i}(aP_i + (d - a)Q_i)$ for any a between 0 and d . Suppose further that*

$$(8.1) \quad g \geq (k')^2 + 2k'(g - 1 - d').$$

Then there exists a nonempty open substack of the ℓ -semistable chain-adaptable locus of $\mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ having the expected dimension $\rho_{\mathcal{L}} := \rho - g + \binom{k}{2}$. If further

$$(g, d, k) \neq (1, 0, 2), (2, 2, 2), (3, 2, 2) \text{ or } (4, 6, 4),$$

the same is true of the ℓ -stable locus.

Proof. The basic strategy is that we consider one set of conditions on the first $(k')^2$ components of X , different conditions on the next $2k'(g - 1 - d')$ components, and finally we allow generic behavior on the remaining $g - (k')^2 - 2k'(g - 1 - d')$ components. On the first set of components, we will impose extra vanishing on two sections per component, with indices cycling over increasing odd numbers of components (that is, we break $(k')^2$ into $1 + 3 + \dots + (2k' + 1)$). On the next collection of components, we impose extra vanishing on only a single section per component, cycling one by one over the $2k'$ sections determining our vector spaces. On the remaining components, no extra vanishing is imposed. See Example A.1. We begin by defining sequences a^i for $i = 1, \dots, g + 1$ as follows:

$$a^1 = 0, 0, 1, 1, \dots, k' - 1, k' - 1, \quad \text{and} \quad a_j^{i+1} = a_j^i + 1 - \epsilon_j^i,$$

with $\epsilon_j^i = 0$ or 1 , and the latter case occurring precisely when one of the following holds:

- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 1$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 1$ or $2m + 1$;
- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 2$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 2$ or $2m + 2$;

- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2m + 1$ for $0 \leq m < k'$, and $j = 2m + 1$ or $2m + 2$;
- $((k')^2 + 1 \leq i \leq (k')^2 + 2k'(g - 1 - d'))$ we have $i = (k')^2 + 2k'm + c$ for some $0 \leq m < (g - 1 - d')$ and $1 \leq c \leq 2k'$, and $j = c$.

Our arguments will make use of the following elementary observations:

- for $i = 1, \dots, g + 1$ and $j = 1, \dots, k' - 1$ we have

$$a_{2j-1}^i \leq a_{2j}^i < a_{2j+1}^i \leq a_{2j+2}^i;$$

- for $i = 1, \dots, g$ and $j = 1, \dots, k'$ we have

$$a_{2j}^{i+1} = a_{2j-1}^i + 1$$

unless $i = (m + 1)^2$ for $0 \leq m < k'$ and $j = m + 1$, in which case $a_{2j}^{i+1} = a_{2j-1}^i$;

- for $i = 1, \dots, (k')^2$, if $j_1 < j_2$ are the indices with $\epsilon_{j_1}^i = \epsilon_{j_2}^i = 1$, we have

$$a_{j_1}^i + a_{j_2}^i = 2(i - 1);$$

- for $i = (k')^2 + 1, \dots, (k')^2 + 2k'(g - 1 - d')$, if j is the index with $\epsilon_j^i = 1$, then

$$2(i - 1) > a_j^i + a_{2k'}^i + 1;$$

- we have

$$a^{g+1} = d', d', d' - 1, d' - 1, \dots, d' - (k' - 1), d' - (k' - 1).$$

We now describe the desired open subset $\mathcal{U} \subseteq \mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ in terms of three conditions, as the locus of tuples $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ satisfying the following:

- (I) each \mathcal{E}^i is semistable on X_i ;
- (II) for $i > (k')^2 + 2k'(g - 1 - d')$, there does not exist any line subbundle \mathcal{L}' of \mathcal{E}^i with either $\mathcal{L}' \cong \mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$ for $a = 0, \dots, d'$, or with $\mathcal{L}'^2 \cong \mathcal{L}_i$;
- (III) for $i = 1, \dots, g - 1$ the vanishing sequence of V_{i+1} at P_{i+1} is equal to a^{i+1} , and the vanishing sequence of V_i at Q_i is equal to $d' - a_k^{i+1}, \dots, d' - a_1^{i+1}$.

Notice that because the imposed vanishing at each Q_i and P_{i+1} has complementary vanishing orders adding to precisely the required d' , to impose the desired vanishing is equivalent to imposing that we have *at most* the desired vanishing, which is an open condition. Thus, our description makes it clear that \mathcal{U} is indeed an open substack of $\mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$, so in order to prove the proposition, we need to show that \mathcal{U} is nonempty of the expected dimension, with the necessary (semi)stability and chain-adaptability properties. The main point will be to give an equivalent, more explicit description of \mathcal{U} . Specifically, we claim that a point $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ of $\mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ is in \mathcal{U} if and only if it satisfies the following conditions:

- (I') the \mathcal{E}^i are described as follows:

- for each $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \oplus \mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i);$$

- for each $i = (k')^2 + 1, \dots, (k')^2 + 2k'(g - 1 - d')$, writing $i = (k')^2 + 2k'm + c$ for $1 \leq c \leq 2k'$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \oplus \mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i);$$

- for each $i > (k')^2 + 2k'(g - 1 - d')$, we have

$$\mathcal{E}^i \cong \mathcal{L}'_i \oplus (\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}),$$

where \mathcal{L}'_i has degree d' , and is chosen so that $(\mathcal{L}'_i)^2 \not\cong \mathcal{L}_i$, and neither \mathcal{L}'_i nor $\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}$ is of the form $\mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$, for $a = 0, \dots, d'$;

(II') for each i , we have

$$V^i = \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)).$$

More precisely, we will show in particular that if \mathcal{E}^i is as in (I'), then the space V^i described in (II') has the correct dimension k , and the vanishing sequences asserted in (III). In fact, in the course of proving the claim, we will further prove that V^i has a (P_i, Q_i) -adapted basis for all i , so that \mathcal{U} consists entirely of chain-adaptable limit linear series. Moreover, this basis may be chosen to come from the maximal summands of \mathcal{E}^i , making gluings easy to analyze.

The first observation is that since $k' \geq 1$, we obtain $d + 2 - g \geq (k')^2$ from (8.1). Thus, for $1 \leq i \leq (k')^2$, we have $\det \mathcal{E}^i \cong \mathcal{O}_{X_i}(2(i - 1)P_i + (d - 2i + 2)Q_i)$. On the other hand, for $i > (k')^2$, if $i \leq d + 2 - g$ we see that

$$\mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i) = \mathcal{O}_{X_i}((2i - 2 - a_c^i) P_i + (d' - 2i + 2 + a_c^i) Q_i).$$

Now, it is clear that (I') implies conditions (I) and (II), and conversely, for $i \leq (k')^2$, condition (I') follows from (I) and (III) by Proposition 8.5. For $(k')^2 + 1 \leq i \leq (k')^2 + 2k'(g - 1 - d')$, we similarly obtain from (I) and (III) that $\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i)$ must be a line subbundle of \mathcal{E}^i , so it suffices to check that \mathcal{E}^i cannot be indecomposable. But this is immediate from the observation that $\mathcal{L}_i \not\cong \mathcal{O}_{X_i}(2a_c^i P_i + 2(d' - a_c^i) Q_i)$, which is a consequence of (iv) above for $i \leq d + 2 - g$ and is a hypothesis for $i > d + 2 - g$. Finally, for $i > (k')^2 + 2k'(g - 1 - d')$, condition (I') is immediate from conditions (I) and (II). It thus suffices to show that assuming the description in condition (I'), we have that (III) is equivalent to (II').

For each $i = 1, \dots, g$ and $j = 1, \dots, k'$, we first observe that it follows from (i) above that

$$\#\{\ell : a_\ell^i \geq a_{2j-1}^i \text{ and } d' - a_\ell^{i+1} \geq d' - a_{2j}^{i+1}\} = \#\{2j - 1, 2j\} = 2.$$

At the same time, if \mathcal{E}^i are as in (I'), we see that

$$\dim \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)) = 2,$$

and this space is generated by a pair of sections s_1 and s_2 , one from each summand of \mathcal{E}^i , with s_1 vanishing to order a_{2j-1}^i at P_i and $d' - a_{2j}^{i+1}$ at Q_i , and s_2 vanishing to order a_{2j}^i at P_i and $d' - a_{2j}^{i+1}$ at Q_i . Indeed, we see immediately from (ii) and the fact that \mathcal{E}^i is a direct sum of two degree- d' line bundles that

$$\dim \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)) \leq 2,$$

with equality except possibly when $i = (m + 1)^2$ for $0 \leq m < k'$. But equality in this last case follows from (I') and (iii), which together with the determinant description imply that $\mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i) \cong \mathcal{O}_{X_i}^{\oplus 2}$. It thus remains to verify that we have sections s_1 and s_2 as described. One checks that for our choice of the a^i , this is equivalent to the condition that \mathcal{E}^i contain as a summand the

bundle $\bigoplus_{\ell: \epsilon_\ell^i=1} \mathcal{O}_{X_i}(a_\ell^i P_i + (d' - a_\ell^i) Q_i)$, with additional summands of the form $\mathcal{O}_{X_i}(a P_i + (d' - a) Q_i)$ only if a is not equal to any a_{2j-1}^i or $a_{2j-1}^i + 1$. This in turn follows for $i \leq (k')^2$ from (iii), and is trivially satisfied for $i > (k')^2 + 2k'(g-1-d')$. In the remaining case, our genericity hypotheses for \mathcal{L} ensures the condition is satisfied if also $i > d+2-g$, and if $i \leq d+2-g$, we conclude the desired statement from (iv).

We thus conclude from Proposition 8.5 that if (III) is satisfied, then V^i contains $\Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i))$. On the other hand, as j varies, (i) implies that these spaces are disjoint, so we conclude that (III) and (I') imply (II'). Note that we have also shown that V^i has a (P_i, Q_i) -adapted basis, giving the asserted chain adaptability. Conversely, we have seen that the V^i described in (II') has vanishing sequences as specified by (III), completing the proof that \mathcal{U} is described equivalently by (I') and (II').

We now investigate gluings. For this, it is useful to observe that the choice of determinant isomorphism ψ is equivalent to a tuple of determinant isomorphisms ψ_i for the restriction to each X_i , commuting with (the determinants of) the gluings maps φ_j . Now, we claim that given bundles \mathcal{E}^i and spaces V^i as specified in (I') and (II'), together with choices of determinant isomorphisms ψ_i , there always exist gluings φ_j , which are unrestricted except for the compatibility with the ψ_i (which can always be achieved by scaling) and the conditions imposed by non-repeated vanishing orders. Indeed, we saw in describing the (P_i, Q_i) -adapted bases of V^i that any nonrepeated vanishing orders are realized by sections in one of the two summands of \mathcal{E}^i , so it is enough to check that when there are nonrepeated vanishing orders they come from summands which can be matched consistently with the summands in the next component. This is easily verified, and we obtain the following explicit conditions imposed on gluings:

- for $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m+1$, if c' is odd and less than $2m+1$ we must have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{c'+1}^{i+1} P_{i+1} - (d' - a_{c'+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}},$$

and

$$\varphi_i \left(\mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i) \Big|_{Q_i} \right) = \mathcal{O}_{X_{i+1}}(a_{c'+1}^{i+1} P_{i+1} + (d' - a_{c'+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}};$$

- for $i = (k')^2 + 1, \dots, (k')^2 + 2k'(g-1-d')$, writing $i = (k')^2 + 2k'm + c$ for $1 \leq c \leq 2k'$, if c is odd we must have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{c+1}^{i+1} P_{i+1} - (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}},$$

and

$$\varphi_i \left(\mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{O}_{X_{i+1}}(a_{c+1}^{i+1} P_{i+1} + (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}}.$$

It then follows in particular that \mathcal{U} is nonempty.

It follows immediately from the definition of \mathcal{U} and Proposition 8.3 that the limit linear series in \mathcal{U} are all ℓ -semistable. Next, observe that due to our genericity hypothesis on the P_i and Q_i , we have that $\mathcal{E}^i \cong \mathcal{L}'^{\oplus 2}$ for some \mathcal{L}' on X_i if and only if $i = m^2$ for some $m \leq k'$. Now, if $k' \geq 2$, there are only two weakly destabilizing line subbundles $\mathcal{M}_3 \subseteq \mathcal{E}^3$, which would have to restrict to one of two lines in the fiber at Q_3 , and hence if we fix a gluing φ_3 , there are only two weakly destabilizing line subbundles $\mathcal{M}_4 \subseteq \mathcal{E}^4$ which could glue to a choice of \mathcal{M}_3 . Thus if

$g > 4$, the unrestricted gluing between \mathcal{E}^4 and \mathcal{E}^5 gives us by Proposition 8.3 that an open dense subset of \mathcal{U} is ℓ -stable. On the other hand, if $k' = 1$ and $g > 3$, we find that we have an unrestricted gluing between \mathcal{E}^3 and \mathcal{E}^4 , which likewise yields ℓ -stability. Similarly, if $k' = 1$, $g - 1 - d' = 0$, and $g = 3$, we obtain ℓ -stability from the unrestricted gluing between \mathcal{E}^2 and \mathcal{E}^3 . The only remaining cases for which $g \geq (k')^2 + 2k'(g - 1 - d')$ are

$$(g, d, k) = (1, 0, 2), (2, 2, 2), (3, 2, 2) \text{ or } (4, 6, 4),$$

as desired.

It remains to compute the dimension of \mathcal{U} . All the \mathcal{E}^i are uniquely determined except when $i > (k')^2 + 2k'(g - 1 - d')$. For the latter, the coarse space for the choices of \mathcal{E}^i is smooth of dimension 1 for each i . Thus, if we consider the forgetful map to the moduli stack of tuples $(\mathcal{E}^i, \psi_i)_{i=1, \dots, g}$ (i.e., the map which forgets gluings and subspaces), we see that the image \mathcal{U}' of \mathcal{U} is a gerbe over a space of dimension $g - ((k')^2 + 2k'(g - 1 - d'))$, with stabilizer group equal to the group of tuples of automorphisms of the \mathcal{E}^i which preserve the ψ_i . The condition of preserving the ψ_i reduces the dimension of each automorphism group by 1, so taking this into account, each \mathcal{E}^i has a 1-dimensional group of automorphisms, except for \mathcal{E}^{a^2} with $a \leq k'$, which each have 3-dimensional automorphism groups. We conclude that the stabilizer groups of \mathcal{U}' have dimension $g + 2k'$, and hence \mathcal{U}' is smooth of dimension

$$g - ((k')^2 + 2k'(g - 1 - d')) - (g + 2k') = -k'(k' + 2g - 2d').$$

The fibers of the forgetful map are described by the choices of gluings and subspaces, but all the V^i are uniquely determined. The gluings also have to commute with the ψ_i , so the dimensions are also each reduced by 1. We thus see from the above description of restrictions on gluing that the dimension of each fiber is

$$3(g - 1) - 2 \cdot \frac{(k')^2 - k'}{2} - 2 \cdot k'(g - 1 - d') = 3g - 3 - k'(k' + 2g - 3 - 2d'),$$

so we conclude that the dimension of \mathcal{U} is

$$\begin{aligned} & -k'(k' + 2g - 2d') + 3g - 3 - k'(k' + 2g - 3 - 2d') \\ & = 3g - 3 - 2k'(2k' + 2g - 2 - 2d') + 2(k')^2 - k' \\ & = \rho_{\mathcal{L}}, \end{aligned}$$

as desired. \square

For the next case, it will be convenient to introduce some additional notation and terminology.

Notation 8.7. Let Y be a smooth, proper curve, and (\mathcal{E}, V) a vector bundle of rank 2 on Y together with a vector space of global sections. Given $P \in Y$, let a_1, \dots, a_k be the vanishing sequence of (\mathcal{E}, V) at P , and suppose that a_i is not repeated in the sequence. Then $\ell(V, a_i, P)$ denotes the line in $\mathcal{E}|_P$ obtained from $V(-aP) \subseteq \Gamma(Y, \mathcal{E}(-aP)) \rightarrow \Gamma(Y, \mathcal{E}(-aP)|_P)$, using the canonical identification $\mathbb{P}(\mathcal{E}|_P) = \mathbb{P}(\mathcal{E}(-aP)|_P)$.

Definition 8.8. Let Y be a smooth, proper curve, $P \in Y$ a point, \mathcal{E} of rank 2 on Y , and $\ell \in \mathcal{E}|_P$ a line. Suppose we have an exact sequence of bundles

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}' \rightarrow 0$$

with $\deg \mathcal{L} = \deg \mathcal{L}'$. If the sequence is split and $\mathcal{L} \not\cong \mathcal{L}'$, we say ℓ is **nondegenerate** if $\ell \neq \mathcal{L}|_P, \mathcal{L}'|_P$. If the sequence is nonsplit, we say ℓ is **nondegenerate** if $\ell \neq \mathcal{L}|_P$.

Note that in either case, the notion of nondegeneracy is intrinsic to \mathcal{E} .

Proposition 8.9. *Assume that $d = 2d'$ is even and $k = 2k' + 1$ is odd, and set $d_i = d$ for $i = 1, \dots, g$. Then the b of Situation 2.1 is determined to be d' . Fix \mathcal{L} as in Proposition 8.2, and assume further that for $i > d + 2 - g$, we have $\mathcal{L}_i \not\cong \mathcal{O}_{X_i}(aP_i + (d - a)Q_i)$ for any a between 0 and d . Suppose further that*

$$(8.2) \quad g \geq (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d').$$

Then there exists a nonempty open substack of the ℓ -semistable chain-adaptable locus of $\mathcal{G}_{2, \mathcal{L}, d}^{k, \text{EHT}}(X)$ having the expected dimension $\rho_{\mathcal{L}} := \rho - g + \binom{k}{2}$. If further $(g, d, k) \neq (3, 4, 3)$, the same is true of the ℓ -stable locus.

Proof. The basic strategy is the same as in Proposition 8.6, except that after the first $(k')^2$ components, we impose a slightly different pattern on the next $k' + 1$ components before continuing as in the previous case. This, combined with the additional section to consider, makes the family of limit linear series more complicated to describe. See Example A.2. We define sequences a^i for $i = 1, \dots, g + 1$ as follows:

$$a^1 = 0, 0, 1, 1, \dots, k' - 1, k' - 1, k' \quad \text{and} \quad a_j^{i+1} = a_j^i + 1 - \epsilon_j^i,$$

with $\epsilon_j^i = 0$ or 1, and the latter case occurring precisely when one of the following holds:

- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 1$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 1$ or $2m + 1$;
- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 2$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 2$ or $2m + 2$;
- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2m + 1$ for $0 \leq m < k'$, and $j = 2m + 1$ or $2m + 2$;
- $((k')^2 + 1 \leq i \leq (k')^2 + k')$ we have $i = (k')^2 + c$ for some $1 \leq c \leq k'$, and $j = 2c - 1$ or $2k' + 1$;
- $(i = (k')^2 + k' + 1)$ we have $j = 2k' + 1$;
- $((k')^2 + k' + 2 \leq i \leq (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d'))$ we have $i = (k')^2 + k' + 1 + (2k' + 1)m + c$ for some $0 \leq m < (g - 1 - d')$ and $1 \leq c \leq 2k' + 1$, and $j = c$.

Let $\mathcal{U} \subseteq \mathcal{G}_{2, \mathcal{L}, d}^{k, \text{EHT}}(X)$ denote the set of $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ satisfying the following:

- (I) each \mathcal{E}^i is semistable on X_i ;
- (II) for $i > (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$, there does not exist any line subbundle \mathcal{L}' of \mathcal{E}^i with either $\mathcal{L}' \cong \mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$ for $a = 0, \dots, d'$, or with $\mathcal{L}'^2 \cong \mathcal{L}_i$;
- (III) for $i = 1, \dots, g - 1$ the vanishing sequence of V_{i+1} at P_{i+1} is equal to a^{i+1} , and the vanishing sequence of V_i at Q_i is equal to $d' - a_k^{i+1}, \dots, d' - a_1^{i+1}$;
- (IV) for $i = (k')^2 + k' + 1$, we have $h^0(X_i, \mathcal{E}^i(-(a_{2k'+1}^i)P_i - (d' - a_{2k'+1}^i)Q_i)) \leq 1$.
- (V) we have $\ell(V_i, a, P_i)$ and $\ell(V_{i-1}, d' - a, Q_{i-1})$ nondegenerate in the following cases:

$$i = m^2 + 2c + 2 \text{ for } 0 \leq m < k', 0 \leq c < m, \text{ and } a = a_{2k'+1}^i,$$

$$i = (k')^2 + c \text{ for } 3 \leq c \leq k' + 1, \text{ and } a = a_1^i,$$

$$i = (k')^2 + k' + 1 + (2k' + 1)m + c \text{ for } 0 \leq m < g - 1 - d', 3 \leq c \leq 2k' + 1 \text{ odd, and } a = a_1^i.$$

As in the proof of Proposition 8.6, we will verify that $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ in $\mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ is in \mathcal{U} if and only if it satisfies the following conditions:

(I') the \mathcal{E}^i are described as follows:

- for each $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \oplus \mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i);$$

- for each $i = (k')^2 + 1, \dots, (k')^2 + k'$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i) Q_i) \oplus \mathcal{L}_i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^i) Q_i);$$

- for $i = (k')^2 + k' + 1$, we have \mathcal{E}^i the unique indecomposable vector bundle of degree d with $\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i) Q_i)$ occurring as a line subbundle;
- for each $i = (k')^2 + k' + 2, \dots, (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$, writing $i = (k')^2 + k' + 1 + (2k' + 1)m + c$ for $1 \leq c \leq 2k' + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \oplus \mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i);$$

- for each $i > (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$, we have

$$\mathcal{E}^i \cong \mathcal{L}'_i \oplus (\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}),$$

where \mathcal{L}'_i has degree d' , and is chosen so that $(\mathcal{L}'_i)^2 \not\cong \mathcal{L}_i$, and neither \mathcal{L}'_i nor $\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}$ is of the form $\mathcal{O}_{X_i}(mP_i + (d' - m)Q_i)$, for $m = 0, \dots, d'$;

(II') The V^i are described as follows:

- for $i \leq (k')^2$, we have

$$V^i = W^i \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i)),$$

where W^i is a 1-dimensional subspace of the 2-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i))$;

- for $i = (k')^2 + 1$, we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i)) \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i));$$

- for $i = (k')^2 + c$, with $2 \leq c \leq k'$, we have

$$\begin{aligned} V^i = W^i \oplus & \left(\bigoplus_{j=1}^{c-2} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1}) Q_i)) \right) \oplus U^i \\ & \oplus \left(\bigoplus_{j=c+1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i)) \right) \\ & \oplus \Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i)), \end{aligned}$$

where W^i is a 1-dimensional subspace of the 2-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1})Q_i))$, and U^i is the (3-dimensional) sum $\Gamma(X_i, \mathcal{E}^i(-a_{2c-2}^i P_i - (d' - a_{2c-1}^{i+1})Q_i)) + \Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1})Q_i))$;

- for $i = (k')^2 + k' + 1$ or $i = (k')^2 + k' + 1 + (2k' + 1)m + c$ with $2 \leq c \leq 2k' + 1$ and $0 \leq m < g - 1 - d'$, or $i > (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$ we have

$$V^i = W^i \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1})Q_i)),$$

where W^i is a 1-dimensional subspace of $\Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1})Q_i))$;

- for $i = (k')^2 + k' + 1 + (2k' + 1)m + 1$ with $0 \leq m < g - 1 - d'$, we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1})Q_i)) \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1})Q_i));$$

(III') the spaces W^i and W^{i-1} are chosen so that $\ell(V^i, a, P_i)$ and $\ell(V^{i-1}, d' - a, Q_{i-1})$ satisfy the nondegeneracy conditions of (V).

We again observe that since $2k' + 1 \geq 2$, we obtain $d + 2 - g \geq (k')^2 + k' + 1$ from (8.2). Thus, for $1 \leq i \leq (k')^2 + k' + 1$, we have $\det \mathcal{E}^i \cong \mathcal{O}_{X_i}(2(i-1)P_i + (d - 2i + 2)Q_i)$. The proof of the equivalence of (I)-(V) with (I')-(III') proceeds as in Proposition 8.6. The only significant difference is that we use condition (IV) to impose indecomposability on \mathcal{E}^i when $i = (k')^2 + k' + 1$. In this case, all global section spaces considered in our description of V^i involve twisting down by degree $d' - 1$, so are 2-dimensional. Note also that for $(k') + 2 \leq i \leq (k')^2 + k'$, the intersection of $\Gamma(X_i, \mathcal{E}^i(-a_{2c-2}^i P_i - (d' - a_{2c-1}^{i+1})Q_i))$ with $\Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1})Q_i))$ is precisely the 1-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1})Q_i))$. We again obtain from the argument that in fact we have chain-adaptability, and – with the exceptions of the sections involved in the lines W^i or the indecomposable case $V^{(k')^2 + k' + 1}$ – that bases may be chosen to lie in summand line bundles.

We next claim that, as in Proposition 8.6, given bundles \mathcal{E}^i and spaces V^i as specified in (I')-(III'), together with choices of determinant isomorphisms ψ_i on each X_i , there always exist gluings φ_j , which are unrestricted except for the conditions imposed by the ψ_i and by non-repeated vanishing orders. This is largely straightforward and similar to the proof of Proposition 8.6, except that one has to check (as can be accomplished by direct calculation) that for the indecomposable vector bundle \mathcal{E}^i with $i = (k')^2 + k' + 1$, we necessarily have $\ell(V^i, a_{2k'}^i, P_i)$ equal to the line at P_i obtained by restricting the line subbundle $\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i)Q_i)$. We obtain the following explicit conditions imposed on gluings:

- for $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we must have

$$\varphi_i(\ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i)) = \ell(V^{i+1}, a_{2k'+1}^{i+1}, P_{i+1})$$

and if c' is odd and less than $2m + 1$ we must also have

$$\varphi_i\left(\mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i)Q_i)\big|_{Q_i}\right) = \mathcal{L}_{i+1}(-a_{c'+1}^{i+1} P_{i+1} - (d' - a_{c'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}},$$

and

$$\varphi_i\left(\mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i)Q_i)\big|_{Q_i}\right) = \mathcal{O}_{X_{i+1}}(a_{c'+1}^{i+1} P_{i+1} + (d' - a_{c'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + 1, \dots, (k')^2 + k'$, we must have

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1})$$

and

$$\varphi_i\left(\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i)Q_i)\big|_{Q_i}\right) = \mathcal{O}_{X_{i+1}}(a_{2k'+1}^{i+1} P_{i+1} + (d' - a_{2k'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + k' + 1$, we have

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \mathcal{O}_{X_{i+1}}(a_1^{i+1} P_{i+1} + (d' - a_1^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + k' + 2, \dots, (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$, writing $i = (k')^2 + k' + 1 + (2k' + 1)m + c$ for some $0 \leq m < g - 1 - d'$ and $1 \leq c \leq 2k' + 1$, we require

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1}),$$

and if c is even we must also have

$$\varphi_i\left(\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i)Q_i)\big|_{Q_i}\right) = \mathcal{L}_{i+1}(-a_{c+1}^{i+1} P_{i+1} - (d' - a_{c+1}^{i+1})Q_{i+1})\big|_{P_{i+1}},$$

and

$$\varphi_i\left(\mathcal{L}_i(-a_c^i P_i - (d' - a_c^i)Q_i)\big|_{Q_i}\right) = \mathcal{O}_{X_{i+1}}(a_{c+1}^{i+1} P_{i+1} + (d' - a_{c+1}^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d') + 1, \dots, g - 1$, we must have

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1}).$$

Note that the gluing of \mathcal{E}^i to \mathcal{E}^{i+1} in the cases $i = (k')^2 + 1$ and $i = (k')^2 + k' + 1 + (2k' + 1)m + 1$ is somewhat special in that $\ell(V^i, d' - a_1^{i+1}, Q_i)$ coincides with the line $\mathcal{L}_i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^i)Q_i)\big|_{Q_i}$ for $i = (k')^2 + 1$ and with the line $\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i)Q_i)\big|_{Q_i}$ for $i = (k')^2 + k' + 1 + (2k' + 1)m + 1$. In the other cases in which it occurs, $\ell(V^i, d' - a_1^{i+1}, Q_i)$ is an independent direction. It again follows in particular that \mathcal{U} is nonempty.

As in Proposition 8.6, we find that the limit linear series in \mathcal{U} are all ℓ -semistable, and that $\mathcal{E}^i \cong \mathcal{L}'^{\oplus 2}$ for some \mathcal{L}' on X_i if and only if $i = a^2$ for some $a \leq k'$. Also as before, we see that if $k' \geq 2$ (in which case $g > 4$) we obtain ℓ -stability on a dense open subset of \mathcal{U} by considering the gluings between \mathcal{E}^3 , \mathcal{E}^4 and \mathcal{E}^5 . On the other hand, if $k' = 1$ and $g > 3$, we find that we have a gluing between \mathcal{E}^3 and \mathcal{E}^4 which does not require gluing together weakly destabilizing line subbundles, likewise yielding ℓ -stability. The only remaining case is $(g, d, k) = (3, 4, 3)$, as desired.

Finally, we compute the dimension of \mathcal{U} as in the proof of Proposition 8.6. Analogously to the previous case, the \mathcal{E}^i are uniquely determined except when $i > (k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')$, and there is a 1-dimensional choice for each in the latter case. Because an indecomposable even-degree vector bundle has automorphism group of dimension 2, we see that the sum of the dimensions of the automorphisms of \mathcal{E}^i preserving the ψ_i is still $g + 2k'$. Thus, if we consider the forgetful map to the moduli stack of tuples $(\mathcal{E}^i, \psi_i)_{i=1, \dots, g}$, we see that the image \mathcal{U}' of \mathcal{U} is a gerbe over a space of dimension $g - ((k')^2 + k' + 1 + (2k' + 1)(g - 1 - d'))$, with stabilizer group of dimension $g + 2k'$, and hence \mathcal{U}' is smooth of dimension

$$g - ((k')^2 + k' + 1 + (2k' + 1)(g - 1 - d')) - (g + 2k') = -(k')^2 - 3k' - 1 - (2k' + 1)(g - 1 - d').$$

The fibers of the forgetful map are described by the choices of gluings and subspaces. The V^i are always either uniquely determined, or have a smooth one-parameter family of choices, and the latter occurs

$$\begin{aligned} & (k')^2 + (k' - 1) + 1 + (2k')(g - 1 - d') \\ & \quad + (g - ((k')^2 + k' + 1) + (2k' + 1)(g - 1 - d')) \\ & = g - 1 - (g - 1 - d') \end{aligned}$$

times. We see from the above description of allowable gluings that, given a fixed choice of the V_i , the space of gluings is smooth of dimension

$$\begin{aligned} & 3(g - 1) - \left(2 \cdot \frac{(k')^2 - k'}{2} + (k')^2 \right) - 2k' - 1 - ((2k' + 1) + 2 \cdot k')(g - 1 - d') \\ & \quad - (g - 1 - ((k')^2 + k' + 1 + (2k' + 1)(g - 1 - d'))) \\ & = 2g - (k')^2 - 2 - 2k'(g - 1 - d'), \end{aligned}$$

so we conclude that the forgetful map is smooth of relative dimension

$$3g - (k')^2 - 3 - (2k' + 1)(g - 1 - d'),$$

and thus that \mathcal{U} is smooth of dimension

$$\begin{aligned} & -(k')^2 - 3k' - 1 - (2k' + 1)(g - 1 - d') + 3g - (k')^2 - 3 - (2k' + 1)(g - 1 - d') \\ & = 3g - 3 - (2k' + 1)(2g - 2 - 2d') - 2(k')^2 - 3k' - 1 \\ & = 3g - 3 - (2k' + 1)((2k' + 1) + 2g - 2 - 2d') + 2(k')^2 + k' \\ & = \rho_{\mathcal{L}}, \end{aligned}$$

as desired. \square

Proposition 8.10. *Assume that $d = 2d' + 1$ is odd and $k = 2k'$ is even, and set $d_{(k')^2 + 2k'(g - 2 - d') + 1} = d$, and $d_i = d - 1$ for $i \neq (k')^2 + 2k'(g - 2 - d') + 1$. Then the b of Situation 2.1 is determined to be d' . Fix \mathcal{L} as in Proposition 8.2, and assume further that for $i > d + 2 - g$, we have $\mathcal{L}_i \not\cong \mathcal{O}_{X_i}(aP_i + (d - 1 - a)Q_i)$ for any a between 0 and $d - 1$. Suppose further that*

$$(8.3) \quad g \geq (k')^2 + 1 + k'(2g - 2 - (2d' + 1)).$$

Then there exists a nonempty open substack of the ℓ -semistable (equivalently, ℓ -stable) chain-adaptable locus of $\mathcal{G}_{2, \mathcal{L}, d}^{k, \text{EHT}}(X)$ having the expected dimension $\rho_{\mathcal{L}} := \rho - g + \binom{k}{2}$.

Proof. The approach is similar to Proposition 8.6, except that we consider an odd-degree indecomposable bundle followed by k' additional specified vector bundles immediately before passing to generic behavior. See Example A.3. We define sequences a^i for $i = 1, \dots, g + 1$ as follows:

$$a^1 = 0, 0, 1, 1, \dots, k' - 1, k' - 1, \quad \text{and} \quad a_j^{i+1} = a_j^i + 1 - \epsilon_j^i,$$

with $\epsilon_j^i = 0$ or 1, and the latter case occurring precisely when one of the following holds:

- ($1 \leq i \leq (k')^2$) we have $i = m^2 + 2c + 1$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 1$ or $2m + 1$;
- ($1 \leq i \leq (k')^2$) we have $i = m^2 + 2c + 2$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 2$ or $2m + 2$;

- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2m + 1$ for $0 \leq m < k'$, and $j = 2m + 1$ or $2m + 2$;
- $((k')^2 + 1 \leq i \leq (k')^2 + 2k'(g - 2 - d'))$ we have $i = (k')^2 + 2k'm + c$ for some $0 \leq m < (g - 2 - d')$ and $1 \leq c \leq 2k'$, and $j = c$;
- $(i = (k')^2 + 2k'(g - 2 - d') + 1)$ we have j odd;
- $((k')^2 + 2k'(g - 2 - d') + 2 \leq i \leq (k')^2 + 2k'(g - 2 - d') + 1 + k')$ we have $i = (k')^2 + 2k'(g - 2 - d') + 1 + c$ with $1 \leq c \leq k'$, and $j = 2c$.

Now, consider the open subset $\mathcal{U} \subseteq \mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ consisting of tuples $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ satisfying the following:

- (I) each \mathcal{E}^i is semistable on X_i ;
- (II) for $i > (k')^2 + 2k'(g - 2 - d') + k' + 1$, there does not exist any line subbundle \mathcal{L}' of \mathcal{E}^i with either $\mathcal{L}' \cong \mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$ for $a = 0, \dots, d'$, or with $\mathcal{L}'^2 \cong \mathcal{L}_i$;
- (III) for $i = 1, \dots, g - 1$ the vanishing sequence of V_{i+1} at P_{i+1} is equal to a^{i+1} , and the vanishing sequence of V_i at Q_i is equal to $d' - a_k^{i+1}, \dots, d' - a_1^{i+1}$;
- (IV) we have $\ell(V^i, a_{2k'}^i, P_i)$ and $\ell(V^i, d' - 1 - a_{2k'}^i, Q_i)$ nondegenerate when $i = (k')^2 + 2k'(g - 2 - d') + 1 + c$ for $1 \leq c \leq k' - 1$, and further $\ell(V^i, d' - a_1^i, Q_i) \neq \ell(V^i, d' - 1 - a_{2k'}^i, Q_i)$ for $i = (k')^2 + 2k'(g - 2 - d') + 1$.

We next claim that a point $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ of $\mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ is in \mathcal{U} if and only if it satisfies the following conditions:

(I') the \mathcal{E}^i are described as follows:

- for each $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \oplus \mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i);$$

- for each $i = (k')^2 + 1, \dots, (k')^2 + 2k'(g - 2 - d')$, writing $i = (k')^2 + 2k'm + c$ for $1 \leq c \leq 2k'$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \oplus \mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i);$$

- for $i = (k')^2 + 2k'(g - 2 - d') + 1$, we have that \mathcal{E}^i is an indecomposable bundle of degree d and determinant \mathcal{L}_i on X_i ;
- for each $i = (k')^2 + 2k'(g - 2 - d') + 2, \dots, (k')^2 + 2k'(g - 2 - d') + k' + 1$, writing $i = (k')^2 + 2k'(g - 2 - d') + 1 + c$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{2c}^i P_i + (d' - a_{2c}^i) Q_i) \oplus \mathcal{L}_i(-a_{2c}^i P_i - (d' - a_{2c}^i) Q_i);$$

- for each $i > (k')^2 + 2k'(g - 2 - d') + k' + 1$, we have

$$\mathcal{E}^i \cong \mathcal{L}'_i \oplus (\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}),$$

where \mathcal{L}'_i has degree d' , and is chosen so that $(\mathcal{L}'_i)^2 \not\cong \mathcal{L}_i$, and neither \mathcal{L}'_i nor $\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}$ is of the form $\mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$, for $a = 0, \dots, d'$;

(II') the spaces V^i are described as follows:

- for $i \leq (k')^2 + 2k'(g - 2 - d')$ and $i > (k')^2 + 2k'(g - 2 - d') + k' + 1$, we have

$$V^i = \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i));$$

- for $i = (k')^2 + k'(g - 2 - d') + 1$ we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_{2k'-1}^{i+1}) Q_i)) + W^i,$$

where W^i is a 2-dimensional subspace of the 3-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_{2k'-1}^i P_i - (d' - a_{2k'}^{i+1}) Q_i))$ which contains $\Gamma(X_i, \mathcal{E}^i(-a_{2k'-1}^i P_i - (d' - a_{2k'-1}^{i+1}) Q_i))$;

- for $i = (k')^2 + 2k'(g - 2 - d') + 1 + c$ with $1 \leq c \leq k'$, we have

$$V^i = \left(\bigoplus_{j=1}^{c-1} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)) \right) \oplus U^i \\ \oplus \left(\bigoplus_{j=c+1}^{k'-1} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1}) Q_i)) \right) \oplus W^i,$$

where W^i is a 1-dimensional subspace of $\Gamma(X_i, \mathcal{E}^i(-a_{2c}^i P_i - (d' - a_{2c+1}^{i+1}) Q_i))$ and U^i is the (3-dimensional) sum

$$\Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1}) Q_i)) + \Gamma(X_i, \mathcal{E}^i(-a_{2c}^i P_i - (d' - a_{2c+1}^{i+1}) Q_i));$$

(III') the spaces W^i are chosen to achieve the nondegeneracy condition of (IV).

This equivalent description of \mathcal{U} is checked as in the previous propositions. In this case, having $k' \geq 1$ implies that $d + 2 - g \geq (k')^2 + 1$, so we have the necessary speciality on the first $(k')^2$ components.

We next consider gluings. As in Proposition 8.6, given bundles \mathcal{E}^i and spaces V^i as specified in (I')-(III'), together with choices of determinant isomorphisms ψ_i , there always exist gluings φ_j , which are unrestricted except for the conditions imposed by the ψ_i and by non-repeated vanishing orders. This imposes the following conditions:

- for $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, if c' is odd and less than $2m + 1$ we must have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{c'+1}^{i+1} P_{i+1} - (d' - a_{c'+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}},$$

and

$$\varphi_i \left(\mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i) \Big|_{Q_i} \right) = \mathcal{O}_{X_{i+1}}(a_{c'+1}^{i+1} P_{i+1} + (d' - a_{c'+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}};$$

- for $i = (k')^2 + 1, \dots, (k')^2 + 2k'(g - 2 - d')$, writing $i = (k')^2 + 2k'm + c$ for $1 \leq c \leq 2k'$, if c is odd we must have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{c+1}^{i+1} P_{i+1} - (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}},$$

and

$$\varphi_i \left(\mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{O}_{X_{i+1}}(a_{c+1}^{i+1} P_{i+1} + (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}};$$

- for $i = (k')^2 + 2k'(g - 2 - d') + 1$, we must have

$$\varphi_i \left(\ell(V^i, d' - a_1^i, Q_i) \right) = \mathcal{L}_{i+1}(-a_2^{i+1} P_{i+1} - (d' - a_2^{i+1}) Q_{i+1}) \Big|_{P_{i+1}}$$

and

$$\varphi_i \left(\ell(V^i, d' - a_{2k'}^{i+1}, Q_i) \right) = \left(\ell(V^{i+1}, a_{2k'}^{i+1}, P_{i+1}) \right);$$

- for $i = (k')^2 + 2k'(g - 2 - d') + 1 + c$ with $1 \leq c \leq k' - 1$, we must have

$$\varphi_i \left(\mathcal{L}_i(-a_{2c}^i P_i - (d' - a_{2c}^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{2(c+1)}^{i+1} P_{i+1} - (d' - a_{2(c+1)}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}}$$

and

$$\varphi_i(\ell(V^i, d' - a_{2k'}^{i+1}, Q_i)) = (\ell(V^{i+1}, a_{2k'}^{i+1}, P_{i+1}));$$

It then follows in particular that \mathcal{U} is nonempty.

It follows immediately from the definition of \mathcal{U} and Proposition 8.3 that the limit linear series in \mathcal{U} are all ℓ -semistable, and since d is odd, this is equivalent to ℓ -stability. It thus remains to compute the dimension of \mathcal{U} . All the \mathcal{E}^i are uniquely determined except when $i = (k')^2 + 2k'(g - 2 - d') + 1$ or $i > (k')^2 + 2k'(g - 2 - d') + k' + 1$. For the former, there are only finitely many choices of an indecomposable bundle of specified determinant, while for the latter, the coarse space for the choices of \mathcal{E}^i is smooth of dimension 1 for each i . Thus, if we consider the forgetful map to the moduli stack of tuples $(\mathcal{E}^i, \psi_i)_{i=1, \dots, g}$ (i.e., the map which forgets gluings and subspaces), we see that the image \mathcal{U}' of \mathcal{U} is a gerbe over a space of dimension $g - ((k')^2 + 2k'(g - 2 - d') + k' + 1)$. The stabilizer group is just as in Proposition 8.6, except that an odd-degree indecomposable vector bundle, being stable, has automorphism group of dimension 1, and hence determinant-fixing automorphism group of dimension 0, so the stabilizer groups of \mathcal{U}' have dimension $g + 2k' - 1$, and hence \mathcal{U}' is smooth of dimension

$$g - ((k')^2 + 2k'(g - 2 - d') + k' + 1) - (g + 2k' - 1) = -k'(k' + 2g - 1 - 2d').$$

The V^i are each either uniquely determined, or have a smooth one-parameter space of choices, with the latter occurring k' times. The choices of gluings have dimension

$$3(g-1) - 2 \cdot \frac{(k')^2 - k'}{2} - 2 \cdot k'(g - 2 - d') - 2 - 2(k' - 1) = 3g - 3 - k'(k' + 2g - 3 - 2d'),$$

so the dimension of each fiber of the forgetful map is

$$3g - 3 - k'(k' + 2g - 3 - 2d') + k' = 3g - 3 - k'(k' + 2g - 4 - 2d'),$$

and we conclude that the dimension of \mathcal{U} is

$$\begin{aligned} & -k'(k' + 2g - 1 - 2d') + 3g - 3 - k'(k' + 2g - 4 - 2d') \\ & = 3g - 3 - 2k'(2k' + 2g - 2 - 2d' - 1) + 2(k')^2 - k' \\ & = \rho_{\mathcal{L}}, \end{aligned}$$

as desired. \square

Proposition 8.11. *Assume that $d = 2d' + 1$ and $k = 2k' + 1$ are odd, and set $d_{(k')^2 + 2k' + 2} = d$, and $d_i = d - 1$ for $i \neq (k')^2 + 2k' + 2$. Then the b of Situation 2.1 is determined to be d' . Fix \mathcal{L} as in Proposition 8.2, and assume further that for $i > d + 2 - g$, we have $\mathcal{L}_i \not\cong \mathcal{O}_{X_i}(aP_i + (d - 1 - a)Q_i)$ for any a between 0 and $d - 1$. Suppose further that*

$$(8.4) \quad g \geq (k')^2 + 1 + (2k' + 1)(g - 1 - d').$$

Then there exists a open substack of the ℓ -semistable (equivalently, ℓ -stable) chain-adaptable locus of $\mathcal{G}_{2, \mathcal{L}, d}^{k, \text{EHT}}(X)$ having the expected dimension $\rho_{\mathcal{L}} := \rho - g + \binom{k}{2}$.

Proof. The construction in this case is a combination of those of Propositions 8.9 and 8.10. The first $(k')^2 + k'$ components are just as in Proposition 8.9, but then the next k' components have specified decomposable bundles with extra vanishing imposed on a single section per component. On the next two components, we take an even-degree indecomposable bundle as in Proposition 8.9, and an odd-degree indecomposable bundle as in Proposition 8.10, and then the next $(2k' + 1)(g - d - 2)$ components are as in the case of Proposition 8.9, before reverting to generic behavior for any remaining components. See Example A.4. We define sequences a^i for $i = 1, \dots, g + 1$ as follows:

$$a^1 = 0, 0, 1, 1, \dots, k' - 1, k' - 1, k' \quad \text{and} \quad a_j^{i+1} = a_j^i + 1 - \epsilon_j^i,$$

with $\epsilon_j^i = 0$ or 1 , and the latter case occurring precisely when one of the following holds:

- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 1$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 1$ or $2m + 1$;
- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2c + 2$ for some $0 \leq m < k'$ and $0 \leq c < m$, and $j = 2c + 2$ or $2m + 2$;
- $(1 \leq i \leq (k')^2)$ we have $i = m^2 + 2m + 1$ for $0 \leq m < k'$, and $j = 2m + 1$ or $2m + 2$;
- $((k')^2 + 1 \leq i \leq (k')^2 + k')$ we have $i = (k')^2 + c$ for some $1 \leq c \leq k'$, and $j = 2c - 1$ or $2k' + 1$;
- $((k')^2 + k' + 1 \leq i \leq (k')^2 + 2k')$ we have $i = (k')^2 + k' + c$ for some $1 \leq c \leq k'$, and $j = 2c$;
- $(i = (k')^2 + 2k' + 1)$ we have $j = 2k' + 1$;
- $(i = (k')^2 + 2k' + 2)$ we have j odd;
- $((k')^2 + 2k' + 3 \leq i \leq (k')^2 + 2k' + 2 + (2k' + 1)(g - 2 - d'))$ we have $i = (k')^2 + 2k' + 2 + (2k' + 1)m + c$ for some $0 \leq m < (g - 2 - d')$ and $1 \leq c \leq (2k' + 1)$, and $j = c$.

Now, consider the open subset $\mathcal{U} \subseteq \mathcal{G}_{2, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X)$ consisting of tuples $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ satisfying the following:

- (I) each \mathcal{E}^i is semistable on X_i ;
- (II) for $i > (k')^2 + 2k' + 2 + (2k' + 1)(g - 2 - d')$, there does not exist any line subbundle \mathcal{L}' of \mathcal{E}^i with either $\mathcal{L}' \cong \mathcal{O}_{X_i}(aP_i + (d' - a)Q_i)$ for $a = 0, \dots, d'$, or with $\mathcal{L}'^2 \cong \mathcal{L}_i$;
- (III) for $i = 1, \dots, g - 1$ the vanishing sequence of V_{i+1} at P_{i+1} is equal to a^{i+1} , and the vanishing sequence of V_i at Q_i is equal to $d' - a_k^{i+1}, \dots, d' - a_1^{i+1}$;
- (IV) for $i = (k')^2 + 2k' + 1$, we have $h^0(X_i, \mathcal{E}^i(-(a_{2k'+1}^i)P_i - (d' - a_{2k'+1}^i)Q_i)) \leq 1$;
- (V) the spaces V^i and V^{i+1} are chosen so that $\ell(V^i, d' - a, Q_i)$ and $\ell(V^{i+1}, a, Q_{i+1})$ are nondegenerate in the following cases:

$$i = m^2 + 2c + 1 \text{ for } 0 \leq m < k', 0 \leq c < m, \text{ and } a = a_{2k'+1}^{i+1},$$

$$i = (k')^2 + c \text{ for } 2 \leq c \leq k', \text{ and } a = a_1^{i+1},$$

$$i = (k')^2 + k' + c \text{ for } 1 \leq c \leq k' - 2, \text{ and } a = a_{2k'+1}^{i+1},$$

$$i = (k')^2 + 2k' + 2 + (2k' + 1)m + c \text{ for } 0 \leq m < (g - 2 - d'), 2 \leq c \leq 2k' \text{ even, and } a = a_1^{i+1}.$$

Also, the spaces V^i are chosen so that $\ell(V^i, d' - a, Q_i)$ is nondegenerate for:

$$i = (k')^2 + k' \text{ and } a = a_1^{i+1},$$

$$i = (k')^2 + 2k' - 1 \text{ and } a = a_{2k'+1}^{i+1},$$

and so that $\ell(V^i, a, Q_i)$ is nondegenerate for:

$$i = (k')^2 + k' + 1 \text{ and } a = a_{2k'+1}^i.$$

We next claim that a point $((\mathcal{E}^i, V^i)_i, (\varphi_j)_j, \psi)$ of $\mathcal{G}_{2, \mathcal{L}, d, \bullet}^{k, \text{EHT}}(X)$ is in \mathcal{U} if and only if it satisfies the following conditions:

(I') the \mathcal{E}^i are described as follows:

- for each $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i) Q_i) \oplus \mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i) Q_i);$$

- for each $i = (k')^2 + 1, \dots, (k')^2 + k'$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i) Q_i) \oplus \mathcal{L}_i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^i) Q_i);$$

- for each $i = (k')^2 + k' + 1, \dots, (k')^2 + 2k'$, writing $i = (k')^2 + k' + c$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_{2c}^i P_i + (d' - a_{2c}^i) Q_i) \oplus \mathcal{L}_i(-a_{2c}^i P_i - (d' - a_{2c}^i) Q_i);$$

- for $i = (k')^2 + 2k' + 1$, we have that \mathcal{E}^i is the unique indecomposable bundle of degree $d - 1$ containing

$$\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i) Q_i)$$

as a line subbundle;

- for $i = (k')^2 + 2k' + 2$, we have that \mathcal{E}^i is an indecomposable bundle of degree d and determinant \mathcal{L}_i on X_i ;
- for each $i = (k')^2 + 2k' + 3, \dots, (k')^2 + 2k' + 2 + (2k' + 1)(g - 2 - d')$, writing $i = (k')^2 + 2k' + 2 + (2k' + 1)m + c$ for $1 \leq c \leq 2k' + 1$, we have

$$\mathcal{E}^i \cong \mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \oplus \mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i);$$

- for each $i > (k')^2 + 2k' + 2 + (2k' + 1)(g - 2 - d')$, we have

$$\mathcal{E}^i \cong \mathcal{L}'_i \oplus (\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}),$$

where \mathcal{L}'_i has degree d' , and is chosen so that $(\mathcal{L}'_i)^2 \not\cong \mathcal{L}_i$, and neither \mathcal{L}'_i nor $\mathcal{L}_i \otimes (\mathcal{L}'_i)^{-1}$ is of the form $\mathcal{O}_{X_i}(a P_i + (d' - a) Q_i)$, for $a = 0, \dots, d'$;

(II') The spaces V^i are described as follows:

- for $i \leq (k')^2$ or $i = (k')^2 + 2k'$, we have

$$V^i = W^i \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i)),$$

where W^i is a 1-dimensional subspace of the 2-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i))$;

- for $i = (k')^2 + 1$ or $i = (k')^2 + 2k' + 1$, we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i)) \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i));$$

- for $i = (k')^2 + c$, with $2 \leq c \leq k'$, we have

$$V^i = W^i \oplus \left(\bigoplus_{j=1}^{c-2} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1}) Q_i)) \right) \oplus U^i \\ \oplus \left(\bigoplus_{j=c+1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)) \right) \\ \oplus \Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i)),$$

where W^i is a 1-dimensional subspace of the 2-dimensional space $\Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1}) Q_i))$, and U^i is the (3-dimensional) sum

$$\Gamma(X_i, \mathcal{E}^i(-a_{2c-2}^i P_i - (d' - a_{2c-1}^{i+1}) Q_i)) + \Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1}) Q_i));$$

- for $i = (k')^2 + k' + c$, with $1 \leq c \leq k' - 1$, we have

$$V^i = \left(\bigoplus_{j=1}^{c-1} \Gamma(X_i, \mathcal{E}^i(-a_{2j-1}^i P_i - (d' - a_{2j}^{i+1}) Q_i)) \right) \oplus U^i \\ \oplus \left(\bigoplus_{j=c+2}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j-2}^i P_i - (d' - a_{2j-1}^{i+1}) Q_i)) \right) \oplus W_1^i \oplus W_2^i,$$

where U^i is the (3-dimensional) sum

$$\Gamma(X_i, \mathcal{E}^i(-a_{2c-1}^i P_i - (d' - a_{2c}^{i+1}) Q_i)) + \Gamma(X_i, \mathcal{E}^i(-a_{2c}^i P_i - (d' - a_{2c+1}^{i+1}) Q_i)),$$

and W_1^i and W_2^i are 1-dimensional subspaces of the 2-dimensional spaces $\Gamma(X_i, \mathcal{E}^i(-a_{2k'}^i P_i - (d' - a_{2k'}^{i+1}) Q_i))$ and $\Gamma(X_i, \mathcal{E}^i(-a_{2k'+1}^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i))$ respectively, chosen to satisfy the condition that $\ell(V^i, a_{2k'}^i P_i) = \ell(V^i, a_{2k'+1}^i P_i)$;

- for $i = (k')^2 + 2k' + 2$ we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_{2k'+1}^{i+1}) Q_i));$$

- for $i = (k')^2 + 2k' + 2 + (2k' + 1)m + 1$ with $0 \leq m < g - 2 - d'$, we have

$$V^i = \Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1}) Q_i)) \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1}) Q_i));$$

- for $i = (k')^2 + 2k' + 2 + (2k' + 1)m + c$ with $2 \leq c \leq 2k' + 1$ and $0 \leq m < g - 2 - d'$, or $i > (k')^2 + 1 + (2k' + 1)(g - 1 - d')$ we have

$$V^i = W^i \oplus \bigoplus_{j=1}^{k'} \Gamma(X_i, \mathcal{E}^i(-a_{2j}^i P_i - (d' - a_{2j+1}^{i+1}) Q_i)),$$

where W^i is a 1-dimensional subspace of $\Gamma(X_i, \mathcal{E}^i(-a_1^i P_i - (d' - a_1^{i+1}) Q_i))$;

(III') the spaces W^i are chosen to achieve the same nondegeneracy conditions as in (V).

This equivalent description of \mathcal{U} is mostly checked as in the previous propositions. The key new points involve the fact that (8.4) implies that $d - g + 2 \geq (k')^2 + 2k' + 1$, so we have $\mathcal{L}_i \cong \mathcal{O}_{X_i}(2(i-1)P_i + 2(d' + 1 - i)Q_i)$ for $i \leq (k')^2 + 2k' + 1$. This is used for $i = (k')^2 + 2k' + 1$, where we have $a_{2k'+1}^i = i - 1$, so (given condition (III)) we have that condition (IV) is equivalent to indecomposability of \mathcal{E}^i . Moreover, we verify that for $i = (k')^2 + k'$ we necessarily have

$$\ell(V^i, d' - a_{2k'}^{i+1}, Q_i) = \ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i),$$

and it also follows from the description of \mathcal{L}_i that for $i = (k')^2 + 2k'$ we necessarily have

$$\ell(V^i, a_{2k'}^i, P_i) = \ell(V^i, a_{2k'+1}^i, P_i).$$

For $(k')^2 + k' < i < (k')^2 + 2k'$, Lemma 4.2 of [Zha14] asserts that because we have \mathcal{E}^i of the form

$$\mathcal{O}_{X_i}(aP_i + (d' - a)Q_i) \oplus \mathcal{O}_{X_i}((2i - 2 - a)P_i + (d' - 2i + 2 + a)Q_i)$$

for a satisfying

$$a < a_{2k'}^i < a_{2k'+1}^i < (2i - 2 - a) - 1,$$

we have $\ell(V^i, a_{2k'}^i, P_i) = \ell(V^i, a_{2k'+1}^i, P_i)$ if and only if $\ell(V^i, d' - a_{2k'}^{i+1}, Q_i) = \ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i)$. The desired description of \mathcal{U} then follows.

We next consider gluings. As in Proposition 8.6, given bundles \mathcal{E}^i and spaces V^i as specified in (I')-(III'), together with choices of determinant isomorphisms ψ_i , there always exist gluings φ_j , which are unrestricted except for the conditions imposed by the ψ_i and by non-repeated vanishing orders. This imposes the following conditions:

- for $i = 1, \dots, (k')^2$, writing $i = m^2 + c'$ for $1 \leq c' \leq 2m + 1$, we must have

$$\varphi_i(\ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i)) = \ell(V^{i+1}, a_{2k'+1}^{i+1}, P_{i+1})$$

and if c' is odd and less than $2m + 1$ we must also have

$$\varphi_i\left(\mathcal{O}_{X_i}(a_{c'}^i P_i + (d' - a_{c'}^i)Q_i)\big|_{Q_i}\right) = \mathcal{L}_{i+1}(-a_{c'+1}^{i+1} P_{i+1} - (d' - a_{c'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}},$$

and

$$\varphi_i\left(\mathcal{L}_i(-a_{c'}^i P_i - (d' - a_{c'}^i)Q_i)\big|_{Q_i}\right) = \mathcal{O}_{X_{i+1}}(a_{c'+1}^{i+1} P_{i+1} + (d' - a_{c'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + 1, \dots, (k')^2 + k' - 1$, we must have

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1})$$

and

$$\varphi_i\left(\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i)Q_i)\big|_{Q_i}\right) = \mathcal{O}_{X_{i+1}}(a_{2k'+1}^{i+1} P_{i+1} + (d' - a_{2k'+1}^{i+1})Q_{i+1})\big|_{P_{i+1}};$$

- for $i = (k')^2 + k'$, we must have

$$\varphi_i(\ell(V^i, d' - a_1^{i+1}, Q_i)) = \left(\mathcal{L}_{i+1}(-a_2^{i+1} P_{i+1} - (d' - a_2^{i+1})Q_{i+1})\big|_{P_{i+1}}\right)$$

and

$$\varphi_i\left(\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i)Q_i)\big|_{Q_i}\right) = \ell(V^{i+1}, a_{2k'+1}^{i+1}, P_{i+1});$$

- for $i = (k')^2 + k' + c$, with $1 \leq c \leq k' - 1$ we must have

$$\varphi_i \left(\mathcal{L}_i(-a_{2c}^i P_i - (d' - a_{2c}^i) Q_i) \Big|_{Q_i} \right) = \left(\mathcal{L}_{i+1}(-a_{2c+2}^{i+1} P_{i+1} - (d' - a_{2c+2}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}} \right)$$

and

$$\varphi_i \left(\ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i) \right) = \ell(V^{i+1}, a_{2k'+1}^{i+1}, P_{i+1});$$

- for $i = (k')^2 + 2k'$, we must have

$$\varphi_i \left(\ell(V^i, d' - a_{2k'+1}^{i+1}, Q_i) \right) = \left(\mathcal{O}_{X_{i+1}}(a_{2k'+1}^{i+1} P_{i+1} + (d' - a_{2k'+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}} \right);$$

- for $i = (k')^2 + 2k' + 1$, we must have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_{2k'+1}^i P_i + (d' - a_{2k'+1}^i) Q_i) \Big|_{Q_i} \right) = \ell(V^{i+1}, a_{2k'+1}^{i+1}, P_{i+1});$$

- for $i = (k')^2 + 2k' + 2$ or $i > (k')^2 + 1 + (2k' + 1)(g - 1 - d')$, we must have

$$\varphi_i \left(\ell(V^i, d' - a_1^{i+1}, Q_i) \right) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1});$$

- for $i = (k')^2 + 2k' + 3, \dots, (k')^2 + 2k' + 2 + (2k' + 1)(g - 2 - d')$, writing $i = (k')^2 + 2k' + 2 + (2k' + 1)m + c$ for some $0 \leq m < g - 2 - d'$ and $1 \leq c \leq 2k' + 1$, we require

$$\varphi_i \left(\ell(V^i, d' - a_1^{i+1}, Q_i) \right) = \ell(V^{i+1}, a_1^{i+1}, P_{i+1}),$$

and if c is even we must also have

$$\varphi_i \left(\mathcal{O}_{X_i}(a_c^i P_i + (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{L}_{i+1}(-a_{c+1}^{i+1} P_{i+1} - (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}},$$

and

$$\varphi_i \left(\mathcal{L}_i(-a_c^i P_i - (d' - a_c^i) Q_i) \Big|_{Q_i} \right) = \mathcal{O}_{X_{i+1}}(a_{c+1}^{i+1} P_{i+1} + (d' - a_{c+1}^{i+1}) Q_{i+1}) \Big|_{P_{i+1}};$$

It then follows in particular that \mathcal{U} is nonempty.

It follows immediately from the definition of \mathcal{U} and Proposition 8.3 that the limit linear series in \mathcal{U} are all ℓ -semistable, and since d is odd, this is equivalent to ℓ -stability. It thus remains to compute the dimension of \mathcal{U} . All the \mathcal{E}^i are uniquely determined except when $i = (k')^2 + 2k' + 2$ or $i > (k')^2 + (2k' + 1)(g - 1 - d') + 1$. For the former, there are only finitely many choices for an indecomposable bundle with specified determinant, while for the latter, the coarse space for the choices of \mathcal{E}^i is smooth of dimension 1 for each i . Thus, if we consider the forgetful map to the moduli stack of tuples $(\mathcal{E}^i, \psi_i)_{i=1, \dots, g}$ (i.e., the map which forgets gluings and subspaces), we see that the image \mathcal{U}' of \mathcal{U} is a gerbe over a space of dimension $g - ((k')^2 + (2k' + 1)(g - 1 - d') + 1)$. The stabilizer group is just as in Proposition 8.10, so the stabilizer groups of \mathcal{U}' have dimension $g + 2k' - 1$, and hence \mathcal{U}' is smooth of dimension

$$g - ((k')^2 + (2k' + 1)(g - 1 - d') + 1) - (g + 2k' - 1) = -(k')^2 - 2k' - (2k' + 1)(g - 1 - d').$$

The V^i are each either uniquely determined, or have smooth one-parameter spaces of choices. The latter occurs

$$(k')^2 + (k' - 1) + (k' - 1) + 1 + (2k')(g - 2 - d') + (g - ((k')^2 + 1 + (2k' + 1)(g - 1 - d'))) = d' - 1$$

times. The choices of gluings have dimension

$$\begin{aligned}
& 3(g-1) - \left(2 \cdot \frac{(k')^2 - k'}{2} + (k')^2 \right) - 2(k'-1) - 2 - 2(k'-1) \\
& \quad - 1 - 1 - 1 - (2k'+1)(g-2-d') - 2 \cdot k'(g-2-d') \\
& \quad - (g-1 - ((k')^2 + 1 + (2k'+1)(g-1-d'))) \\
& \quad = 3g - 3 - (k')^2 - (2k'+1)(g-2-d') - d' - k',
\end{aligned}$$

so the dimension of each fiber of the forgetful map is

$$\begin{aligned}
& 3g - 3 - (k')^2 - (2k'+1)(g-2-d') - d' - k' + d' - 1 \\
& \quad = 3g - 3 - (k')^2 - (2k'+1)(g-2-d') - k' - 1,
\end{aligned}$$

and we conclude that the dimension of \mathcal{U} is

$$\begin{aligned}
& -(k')^2 - 2k' - (2k'+1)(g-1-d') + 3g - 3 - (k')^2 - (2k'+1)(g-2-d') - k' - 1 \\
& \quad = 3g - 3 - (2k'+1)(2g-2-2d'-1) - 2(k')^2 - 3k' - 1 \\
& \quad = \rho_{\mathcal{L}},
\end{aligned}$$

as desired. \square

We now easily complete the proof of our second main theorem.

Proof of Theorem 1.2. In light of Proposition 8.2, this follows almost immediately from Theorem 1.1 and Propositions 8.6, 8.9, 8.10 and 8.11. Indeed, the inequality of the theorem statement in the even (respectively, odd) degree case is the same as that in Proposition 8.6 (respectively, Proposition 8.10), and the equivalence of the inequality of Proposition 8.9 (respectively, Proposition 8.11) follows from the integrality of the quantities involved. For the stability assertion, the only remaining case is $(g, d, k) = (3, 4, 3)$, which can be proved by direct analysis on smooth curves – see §3 of [BF98] or page 105 of [Tei04]. \square

9. FURTHER DISCUSSION

We begin by discussing the ranges of g, k, d for which Theorem 1.2 and Corollary 1.3 apply.

Remark 9.1. In some sense, the range $g-2 \leq d \leq 2g-2$ is misleading, in that in fact for any given g , the degree cannot be too close to $g-2$ without violating the main inequalities of Theorem 1.2. For instance, if $d = g-2$, these inequalities are never satisfied with $k \geq 2$, and if $d = g-1$, the only case which occurs is $k = 2$ and g odd. Despite these limitations, if we set $m = -\chi(\mathcal{L}) = 2g-2-d$, then for any fixed value of $m \geq 0$, for sufficiently large g we obtain (increasingly large) ranges of k for which both Theorem 1.2 and Corollary 1.3 apply, and similarly, for sufficiently large k we obtain (increasingly large) ranges of g for which both results apply. In particular, we have produced a large infinite family of examples of components of the stable locus of $G_{2,d}^k(X)$ having strictly larger than the expected dimension $\rho-1$.

We also mention that, while our existence results are certainly not optimal, and are extended by Zhang in [Zha14] using our smoothing theorem, unlike the canonical determinant case one should not expect the existence results of Theorem 1.2 to extend to all cases for which $\rho_{\mathcal{L}} \geq 0$. Indeed, when $d = g-2$ and $k = 2$, we have $\rho_{\mathcal{L}} = g-6 \geq 0$ for $g \geq 6$, but in this case, the locus of stable $\mathfrak{g}_{2,d}^2$ s

is supported over determinants \mathcal{L} with $h^1(\mathcal{L}) > 1$; see Example 6.1 of [Oss13a]. Thus, if $h^1(\mathcal{L}) = 1$ and $d = g - 2$, there are no stable $\mathfrak{g}_{2,d}^k$ s with determinant \mathcal{L} for any $k \geq 2$. Accordingly, the limitations of our results in the case that d is close to $g - 2$ are due in part to actual failure of existence in this range. There is no evidence against the possibility that, by studying more complicated families of limit linear series, the machinery of Theorem 1.1 could eventually produce sharp existence statements for the case that $h^1(\mathcal{L}) = 1$.

We next discuss cases of Theorem 1.2 in which $\rho_{\mathcal{L}}$ is negative.

Remark 9.2. In Theorem 1.2, the cases $(g, d, k) = (1, 0, 2), (3, 2, 2)$ or $(4, 6, 4)$ have $\rho_{\mathcal{L}} < 0$; in fact, it is elementary that these are precisely the cases for which the hypothesized inequality is satisfied, but $\rho_{\mathcal{L}} < 0$. This does not lead to a contradiction, because in these cases we are only claiming to produce semistable limit linear series, which by virtue of larger automorphism groups may belong to a negative-dimensional family.

Finally, for the sake of broader context we briefly discuss an observation of Gregorczyk and Newstead.

Remark 9.3. Even restricting to the case $h^1(\mathcal{L}) = 1$, the picture afforded by our modified expected dimension $\rho_{\mathcal{L}}$ is incomplete, in the sense that we can have the stable locus of $G_{2,\mathcal{L}}^k(X)$ nonempty for X general even when $\rho_{\mathcal{L}} < 0$. Indeed, Farkas and Ortega studied the case $g = 2a + 1$, $d = 2a + 4$, and $k = 4$ in [FO11], and found that the locus of stable $\mathfrak{g}_{2,d}^k$ s is supported over determinants \mathcal{L} with $h^1(\mathcal{L}) = 1$. However, Grzegorzczuk and Newstead observed in Example 4.2 of [GN] that in this case $\rho_{\mathcal{L}} = 7 - g$, which is negative for $g > 7$. Farkas and Ortega actually show that the $\mathfrak{g}_{2,d}^k$ s are supported over a smaller locus of determinants satisfying an additional Koszul condition, demonstrating that in order to understand the behavior of the fixed determinant case, it is necessary to consider invariants subtler than h^1 . Interestingly, in the case studied by Farkas and Ortega, the total dimension (for varying determinant) still works out as predicted by our $\rho_{\mathcal{L}}$; see Example 5.2 of [Oss13c].

APPENDIX A. EXAMPLES

As a companion to the rather complicated descriptions of the families of limit linear series described in §8, we give small examples for each of the four cases considered. On each component, we list the vanishing sequences at P_i and Q_i , as well as the underlying vector bundles. We use $\mathcal{O}(a, b)$ as shorthand for $\mathcal{O}_{X_i}(aP_i + bQ_i)$, and if the determinant of \mathcal{E}^i is fixed to be \mathcal{L}_i , we similarly write $\mathcal{L}_i(a, b)$ for $\mathcal{L}_i(-aP_i - bQ_i)$. We also use $\text{EI}(a, b)$ to denote the unique (even-degree) indecomposable vector bundle of rank 2 and degree $2a + 2b$ containing $\mathcal{O}(a, b)$ as a line subbundle, and $\text{OI}(\mathcal{L}_i)$ to denote an (odd-degree) indecomposable vector bundle of rank 2 and determinant \mathcal{L}_i .

All cases follow the same pattern on the first $(k')^2$ components, and on the final components when g is strictly greater than required. In addition, when $d < 2g - 3$, each case involves an interval of cycling through all k sections in succession. In order to keep examples at a manageable size, we keep g minimal in each case, and we only have $d < 2g - 3$ in the first example.

0	6	0	6	0	5	1	4	2	4	2	3	3	2	4	1
0	6	0	5	1	5	1	4	2	3	3	3	3	2	4	1
1	4	2	4	2	3	3	3	3	2	4	1	5	1	5	0
1	4	2	3	3	3	3	3	3	2	4	1	5	0	6	0
$\mathcal{O}(0, 6)$	$\mathcal{O}(0, 6)$	$\mathcal{O}(1, 5)$	$\mathcal{O}(3, 3)$	$\mathcal{O}(2, 4)$	$\mathcal{O}(3, 3)$	$\mathcal{O}(5, 1)$	$\mathcal{O}(6, 0)$								
\oplus	\oplus	\oplus	\oplus	\oplus	\oplus	\oplus	\oplus								
$\mathcal{O}(0, 6)$	$\mathcal{O}(2, 4)$	$\mathcal{O}(3, 3)$	$\mathcal{O}(3, 3)$	$\mathcal{O}(6, 0)$	$\mathcal{O}(7, -1)$	$\mathcal{L}_7(5, 1)$	$\mathcal{L}_8(6, 0)$								

0	6	0	6	0	5	1	4	2	4	2	3	3	2		
0	6	0	5	1	5	1	4	2	3	3	2	4	1		
1	4	2	4	2	3	3	3	3	2	4	2	4	1		
1	4	2	3	3	3	3	3	3	2	4	1	5	0		
2	3	3	2	4	1	5	0	6	0	6	0	6	0		
$\mathcal{O}(0,6)$		$\mathcal{O}(0,6)$		$\mathcal{O}(1,5)$		$\mathcal{O}(3,3)$		$\mathcal{O}(2,4)$		$\mathcal{O}(4,2)$		EI(6,0)			
\oplus		\oplus		\oplus		\oplus		\oplus		\oplus					
$\mathcal{O}(0,6)$		$\mathcal{O}(2,4)$		$\mathcal{O}(3,3)$		$\mathcal{O}(3,3)$		$\mathcal{O}(6,0)$		$\mathcal{O}(6,0)$					

0	5	0	5	0	4	1	3	2	3	2	2	3	1
0	5	0	4	1	4	1	3	2	2	3	2	3	1
1	3	2	3	2	2	3	2	3	2	3	1	4	0
1	3	2	2	3	2	3	2	3	1	4	0	5	0
$\mathcal{O}(0,5)$		$\mathcal{O}(0,5)$		$\mathcal{O}(1,4)$		$\mathcal{O}(3,2)$		$\text{OI}(\mathcal{O}(8,3))$			$\mathcal{O}(3,2)$		$\mathcal{O}(5,0)$
\oplus		\oplus		\oplus		\oplus					\oplus		\oplus
$\mathcal{O}(0,5)$		$\mathcal{O}(2,3)$		$\mathcal{O}(3,2)$		$\mathcal{O}(3,2)$					$\mathcal{O}(6,-1)$		$\mathcal{L}_7(5,0)$

0	8	0	8	0	7	1	6	2	6	2	5	3	4	4	3	5	2	6	2
0	8	0	7	1	7	1	6	2	5	3	4	4	4	4	3	5	2	6	1
1	6	2	6	2	5	3	5	3	4	4	4	4	3	5	2	6	1	7	1
1	6	2	5	3	5	3	5	3	4	4	3	5	2	6	2	6	1	7	0
2	5	3	4	4	3	5	2	6	2	6	2	6	1	7	0	8	0	8	0
$\vartheta(0, 8)$ \oplus $\vartheta(0, 8)$	$\vartheta(0, 8)$ \oplus $\vartheta(2, 6)$	$\vartheta(1, 7)$ \oplus $\vartheta(3, 5)$	$\vartheta(3, 5)$ \oplus $\vartheta(3, 5)$	$\vartheta(2, 6)$ \oplus $\vartheta(6, 2)$	$\vartheta(4, 4)$ \oplus $\vartheta(6, 2)$	$\vartheta(4, 4)$ \oplus $\vartheta(8, 0)$	$\vartheta(6, 2)$ \oplus $\vartheta(8, 0)$	$\text{EI}(8, 0) \quad \text{OI}(\mathcal{L}_{10})$											

Note that \mathcal{L}_{10} has odd degree 17 in this case.

INDEX OF NOTATION AND TERMINOLOGY

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